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MITROVIC'S METHOD - SOME FUNDAMENTAL TECHNIQUES.

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MITROVIC'S METHOD -

SOME FUNDAMENTAL TECHNIQUES

by

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CHAPTER I

BASIC CONCEPTS

1.1 Introduction, Objective

In the analysis of linear dynamic systems the mathematical description can be formulated as a single ordinary linear differential equation with constant coefficients. Upon application of transform methods a polynomial commonly called the "characteristic equation" is obtained. An essential part of most analysis problems is the evaluation of the roots (zeros) of this characteristic equation. Mitrovic (1,2) has developed a graphical technique which permits ready evaluation of the roots as a function of the values of two of the coefficients of the polynomial.

While there are many methods for evaluating the roots of a polynomial, few of these methods are of significant advantage in engineering synthesis problems. Mitrovic's method, because it relates the root values to coefficient values in a convenient graphical representation, provides a very useful approach to design.

Most of the methods available for analysis and design are either <u>one parameter</u> methods (such as the root locus method) or evaluate dynamic characteristics indirectly, or both (such as most frequency domain methods). Mitrovic's method is basically a <u>two parameter</u> method, but is readily extended to <u>three parameters</u>. It is possible to extend the method to handle <u>more</u> than three parameters, but the practical value of such techniques has not yet been established.

The original development by Mitrovic (1,2) considered only two coefficients of the polynomial, i.e., for an expression of the form

$$a_n s^n + a_{n-1} s^{n-1} + - - - a_3 s^3 + a_2 s^2 + a_1 s + a_o = 0 \qquad (1\text{-}1)$$

the original development considers analysis and synthesis only in terms of a_o and a_1. More recently (3,4) the basic concepts have been generalized to permit analysis and design in terms of any two coefficients. The objective of this research paper is to provide detailed analyses procedures, equations and techniques which will reduce the more recent advances to a more practical level. This is to be done by providing:

a) detailed stability analyses and rules for applying them to any of the Mitrovic planes.

b) tabulated equation pairs to assist the computations.

c) procedures and techniques for sketching the curves with a minimum amount of computational labor.

d) tabulated relationships to aid the sketching.

e) a digital computer program to assist with detailed computations.

f) some selected illustrations of the results in design applications.

1-2. Review of Mitrovic's Method.

If all of the roots of a polynomial be inside some area in the s-plane, then proof of this can be established by enclosing the area by a contour, mapping the contour onto a polar plane through the characteristic polynomial as a mapping function, and analyzing the polar contour with the Principle of Argument. This is essentially Mitrovic's Method. He chose as mapping contours only the imaginary axis or a radial straight line in the left half plane, closing his contour through a circular arc of infinite radius so as to enclose all or part of the left half plane.

With mapping contour defined as above, a point on the contour is defined by

$$s = \omega_n e^{j\left(\frac{\pi}{2} + \theta\right)} = -\omega_n \sin\theta + j\,\omega_n \cos\theta$$
$$= -\zeta\,\omega_n + j\,\omega_n\sqrt{1 - \zeta^2} \qquad (1\text{-}2)$$

Inserting equation 1-2 in 1-1 and requiring that reals and imaginaries are zero independently provides two equations in terms of ω_n, the coefficients, and certain functions of ζ. But ζ is a known number and for the usual case of analysis the coefficients are known. Thus two equations in one unknown (ω_n) are obtained. Mitrovic's procedure was to define (select) two coefficients as unknowns, thus providing two equations in two unknowns and a common parameter, ω_n. He solved the equations for the unknown coefficients in terms of ω_n. Then choosing a value for ζ and letting ω_n take selected values, curves can be obtained by plotting one coefficient as ordinate and the second as abscissa with ω_n as a parameter.

If this is done for a polynomial for which all coefficients are known numerically, then the actual values of the coefficients which were designated variables are the coordinates of a single point on the Mitrovic plot. Mitrovic showed how to evaluate stability and all left half plane roots of the polynomial from his curves and the location of this one point. If this point, called the M-point, is moved to a new location new roots and new coefficient values are defined. Then the physical system can be provided with the dynamics specified by the new roots if the system can be changed so that the two designated coefficients assume their new values without changing any other coefficients. This, in essence, is the result provided by Mitrovic, and to it he added considerable detail regarding specific techniques for using the method to design feedback control systems.

In his treatment Mitrovic defined as variables only the coefficients a_1 and a_o (see equation 1-1). Elliott, Heseltine and Thaler (3) used several other coefficients as variables in specific problems involving the design of feedback compensation for control systems, but did not generalize these results. Later Siljak made an independent study which generalized Mitrovic's results to any pair of coefficients, but did not expand or apply his results. This, then, was essentially the state of development of Mitrovic's method when this research was undertaken; the basic method using only the a_o and a_1 coefficients is well developed and techniques are available for applying it to many design problems. Mitrovic (1) and Siljak (5) also applied this method to sampled data systems. The basic theory of the generalized method has been established, and a few techniques have been developed to apply it to feedback compensation design. In general, practical ways to interpret stability, and to apply the generalized theory to analysis and design are lacking. Some advances in these areas are presented in this report.

1-3 Bibliography.

1. MITROVIC, DUSAN Graphical Analysis and Synthesis of Feedback Control Systems.

   PART I Theory and Analysis.

   PART II Synthesis.

   PART III Sampled Data Feedback Control Systems.

   Trans AIEE Pt. II January 1959

2. THALER G. J. and BROWN R. G. Analysis and Design of Feedback Control Systems, Ch. 10.

   McGraw-Hill, 1960

3. ELLIOTT D. W., G. J. THALER and J. C. W. HESELTINE, Feedback Compensation Using Derivative Signals.

PART I Routh's Criterion, Root Loci

PART II Mitrovic's Method

Trans IEEE, Applications and Industry, November 1963.

4. SILJAK, D. Generalized Mitrovic's Method.

Trans IEEE, Applications and Industry (to be published).

5. SILJAK, D. Sampled Data Systems with Transport Lag by Mitrovic's Algebraic Method.

Trans AIEE Pt. II, November 1961.

6. MITROVIC D. M. Automatic Plotting of Characteristic Curves and Analogue Solution of Algebraic Equations

Trans AIEE Pt. II

CHAPTER II

THE GENERALIZED METHOD

2.1 Siljak's Results.

Through a formal derivation and rigorous proof Siljak obtained, in symbolic form, a generalized expression for the Mitrovic Curves in terms of any two selected coefficients. These results are; for Bi = Bj, (i > j)

$$Bi = \left[\sum_{k=0}^{N} A_k\, \omega_n^k\, \varphi_{k-j}\ (\zeta)\right] \Big/ \left[\omega_n^i\, \varphi_{j-i}\ (\zeta)\right]$$

$$Bj = \left[\sum_{k=0}^{N} A_k\, \omega_n^k\, \varphi_{k-i}\ (\zeta)\right] \Big/ \left[\omega_n^j\, \varphi_{i-j}\ (\zeta)\right]$$

where A_k is the coefficient of the k'th order term of the original polynomial, and $\varphi_k(\zeta)$ are the functions defined by Mitrovic.

2.2 The Determinantal Approach. Tabulation of Equation Pairs.

Mitrovic's original result, (rearranged as in Ref. 2) expressed with all coefficients designated as a's is:

$$a_o = -\omega_n^2 \left[a_2\, \varphi_1 + a_3\, \omega_n\, \varphi_2 + a_4\, \omega_n^2\, \varphi_3 + \ldots + a_n\, \omega_n^{n-2}\, \varphi_{n-1} \ldots\right]$$

$$a_1 = a_2\, \varphi_2\, \omega_n + a_3\, \varphi_3\, \omega_n^2 + a_4\, \varphi_4\, \omega_n^3 + \ldots + a_n\, \varphi_n\, \omega_n^{n-1}.$$

where the φ's are functions of ζ and are tabulated elsewhere in this report.

In the original method a_o and a_1 were designated variables. In general any two coefficients may be chosen. To obtain specific equations for any two coefficients rearrange the above equations with both of the chosen coefficients on the left hand side; for example a_2 and a_3:

$$a_2\, \omega_n^2\, \varphi_1 + a_3\, \omega_n^3\, \varphi_2 = -a_o - \omega_n^2 \left[a_4\, \omega_n^2\, \varphi_3 + \ldots + a_n\, \omega_n^{n-2}\, \varphi_{n-1}\right]$$

$$-a_2\, \varphi_2\, \omega_n - a_3\, \varphi_3\, \omega_n^2 = -a_1 + a_4\, \varphi_4\, \omega_n^3 + \ldots + a_n\, \varphi_n\, \omega_n^{n-1}$$

These equations are simply simultaneous equations in a_2 and a_3, and may be solved by the usual determinantal methods, i.e.,

$$a_2 = \frac{\Delta_2}{\Delta}. \quad a_3 = \frac{\Delta_3}{\Delta}$$

where $$\Delta = \begin{bmatrix} \omega_n^2 \varphi_1 & \omega_n^3 \varphi_2 \\ -\omega_n \varphi_2 & -\omega_n^2 \varphi_3 \end{bmatrix}$$

$$\Delta_2 = \begin{bmatrix} -a_o - \omega_n^2 \left[a_4 \omega_n^2 \varphi_3 + \ldots + a_n \omega_n^{n-2} \varphi_{n-1} \right] & \omega_n^3 \varphi_2 \\ -a_1 + a_4 \varphi_4 \omega_n^3 + \ldots + a_n \varphi_n \omega_n^{n-1} & -\omega_n^2 \varphi_3 \end{bmatrix}$$

$$\Delta_3 = \begin{bmatrix} \omega_n^2 \varphi_1 & -a_o - \omega_n^2 \left[a_4 \omega_n^2 \varphi_3 + \ldots a_n \omega_n^{n-2} \varphi_{n-1} \right] \\ -\omega_n \varphi_2 & -a_1 + a_4 \varphi_4 \omega_n^3 + \ldots + a_n \varphi_n \omega_n^{n-1} \end{bmatrix}$$

If the indicated algebra is performed the desired relationships are obtained, as tabulated in Table 2-1 which follows:

TABLE I

(1) $B_o - B_1$

$$B_o = \left[A_2 \omega_n^2 \varphi_1 (\zeta) + A_3 \omega_n^3 \varphi_2 (\zeta) + \ldots + A_n \omega_n^n \varphi_{n-1} (\zeta) \right]$$

$$B_1 = A_2 \omega_n \varphi_2 (\zeta) + A_3 \omega_n^2 \varphi_3 (\zeta) + \ldots + A_n \omega_n^{n-1} \varphi_n (\zeta)$$

(2) $B_o - B_2$ *

$$B_o = \varphi_2^{-1} (\zeta) \left[+ A_1 \omega_n \varphi_1 (\zeta) + A_3 \omega_n^3 \varphi_1 (\zeta) + A_4 \omega_n^4 \varphi_2 (\zeta) + \ldots A_n \omega_n^n \varphi_{n-2} (\zeta) \right]$$

$$B_2 = \omega_n^{-2} \varphi_2^{-1} (\zeta) \left[-A_1 \omega_n \varphi_1 (\zeta) - A_3 \omega_n^3 \varphi_3 (\zeta) - A_4 \omega_n^4 \varphi_4 (\zeta) - \ldots - A_n \omega_n^n \varphi_n (\zeta) \right]$$

(3) $B_o - B_3$ **

$$B_o = \varphi_3^{-1} (\zeta) \left[- A_1 \omega_n \varphi_2 (\zeta) - A_2 \omega_n^2 \varphi_1 (\zeta) + A_4 \omega_n^4 \varphi_1 (\zeta) + \ldots + A_n \omega_n^n \varphi_{n-3} (\zeta) \right]$$

$$B_3 = -\omega_n^{-3} \varphi_3^{-1} (\zeta) \left[+ A_1 \omega_n \varphi_1 (\zeta) + A_2 \omega_n^2 \varphi_2 (\zeta) + A_4 \omega_n^4 \varphi_4 (\zeta) + \ldots + A_n \omega_n^n \varphi_n (\zeta) \right]$$

(4) $B_o - B_4$ *

$$B_o = \varphi_4^{-1} (\zeta) \left[A_1 \omega_n \varphi_3 (\zeta) + A_2 \omega_n^2 \varphi_2 (\zeta) + A_3 \omega_n^3 \varphi_1 (\zeta) - A_5 \omega_n^5 \varphi_1 (\zeta) - \ldots A_n \omega_n^n \varphi_{n-4} (\zeta) \right]$$

$$B_4 = -\omega_n^{-4}\,\varphi_4^{-1}\,(\zeta)\left[+ A_1\,\omega_n\,\varphi_1\,(\zeta) + A_2\,\omega_n^2\,\varphi_2\,(\zeta) + A_3\,\omega_n^3\,\varphi_3\,(\zeta) + A_5\,\omega_n^5\,\varphi_5\,(\zeta) + \ldots + A_n\,\omega_n^n\,\varphi_n\,(\zeta)\right]$$

(5) $B_1 - B_2$

$$B_1 = -\,\omega_n^{-1}\left[- A_o\,\varphi_2\,(\zeta) + A_3\,\omega_n^3\,\varphi_1\,(\zeta) + A_4\,\omega_n^4\,\varphi_2\,(\zeta) + \ldots + A_n\,\omega_n^n\,\varphi_{n-2}\,(\zeta)\right]$$

$$B_2 = \omega_n^{-2}\left[- A_o\,\varphi_1\,(\zeta) + A_3\,\omega_n^3\,\varphi_2\,(\zeta) + \ldots + A_n\,\omega_n^n\,\varphi_{n-1}\,(\zeta)\right]$$

(6) $B_1 - B_3$*

$$B_1 = \omega_n^{-1}\,\varphi_2^{-1}\,(\zeta)\left[- A_o\,\varphi_3\,(\zeta) - A_2\,\omega_n^2\,\varphi_1\,(\zeta) + A_4\,\omega_n^4\,\varphi_1\,(\zeta) + A_n\,\omega_n^n\,\varphi_{n-3}(\zeta)\right]$$

$$B_3 = -\omega_n^{-3}\,\varphi_2^{-1}\,(\zeta)\left[A_o + A_2\,\omega_n^2\,\varphi_1\,(\zeta) + A_4\,\omega_n^4\,\varphi_3\,(\zeta) + \ldots + A_n\,\omega_n^n\,\varphi_{n-1}\,(\zeta)\right]$$

(7) $B_1 - B_4$**

$$B_1 = \omega_n^{-1}\,\varphi_3^{-1}\,(\zeta)\left[-A_o\,\varphi_4\,(\zeta) - A_2\omega_n^2\,\varphi_2(\zeta) - A_3\,\omega_n^3\,\varphi_1(\zeta) + A_5\,\omega_n^5\,\varphi_1\,(\zeta) + \ldots + A_n\omega_n^n\,\varphi_{n-4}\,(\zeta)\right]$$

$$B_4 = -\omega_n^{-4}\,\varphi_3^{-1}\,(\zeta)\left[A_o + A_2\,\omega_n^2\,\varphi_1\,(\zeta) + A_3\,\omega_n^3\,\varphi_2\,(\zeta) + A_5\,\omega_n^5\,\varphi_4(\zeta) + \ldots + A_n\,\omega_n^n\,\varphi_{n-1}\,(\zeta)\right]$$

(8) $B_2 - B_3$

$$B_2 = -\omega_n^{-2}\left[- A_o \varphi_3(\zeta) - A_1 \omega_n \varphi_2(\zeta) + A_4 \omega_n^4 \varphi_1(\zeta) + A_5 \omega_n^5 \varphi_2(\zeta) + \ldots + A_n \omega_n^n \varphi_{n-3}(\zeta)\right]$$

$$B_3 = \omega_n^{-3}\left[- A_o \varphi_2(\zeta) - A_1 \omega_n \varphi_1(\zeta) + A_4 \omega_n^4 \varphi_2(\zeta) + \ldots + A_n \omega_n^n \varphi_{n-2}(\zeta)\right]$$

(9) $B_2 - B_4$*

$$B_2 = \omega_n^{-2} \varphi_2^{-1}(\zeta)\left[- A_o \varphi_4(\zeta) - A_1 \omega_n \varphi_3(\zeta) - A_3 \omega_n^3 \varphi_1(\zeta) + A_5 \omega_n^5 \varphi_1 + \ldots + A_n \omega_n^n \varphi_{n-4}(\zeta)\right]$$

$$B_4 = -\omega_n^{-4} \varphi_2^{-1}(\zeta)\left[-A_o \varphi_2(\zeta) - A_1 \omega_n \varphi_1(\zeta) + A_3 \omega_n^3 \varphi_1(\zeta) + A_5 \omega_n^5 \varphi_3(\zeta) + \ldots + A_n \omega_n^n \varphi_{n-2}(\zeta)\right]$$

(10) $B_3 - B_4$

$$B_3 = \omega_n^{-3} \varphi_1^{-1}(\zeta)\left[- A_o \varphi_4(\zeta) - A_1 \omega_n \varphi_3(\zeta) - A_2 \omega_n^2 \varphi_2(\zeta) + A_5 \omega_n^5 \varphi_1(\zeta) + \ldots + A_n \omega_n^n \varphi_{n-4}(\zeta)\right]$$

$$B_4 = -\omega_n^{-4} \varphi_1^{-1}(\zeta)\left[- A_o \varphi_3(\zeta) - A_1 \omega_n \varphi_2(\zeta) - A_2 \omega_n^2 \varphi_1(\zeta) + A_5 \omega_n^5 \varphi_2(\zeta) + \ldots + A_n \omega_n^n \varphi_{n-3}(\zeta)\right]$$

Note: Those with * and ** have singularities at $\zeta = 0$ and $\zeta = .5$ respectively.

2.3 Discussion.

The equations of Table I are all of essentially the same form, and may be plotted or sketched in essentially the same way. Interpretation of the results is not obvious, however. Certainly if the imaginary axis is mapped ($\zeta = 0$) it should be possible to define stability from the location of the M-point, but stability has been defined pictorially only for the B_o vs B_1 case. Thus the interpretation of stability and other related topics must be investigated, and this is reported in Chapter 3.

One immediate difficulty appears in the equations for even-even or odd-odd coefficients, such as $B_o - B_2$; $B_2 - B_4$; $B_1 - B_3$; $B_o - B_4$. All such equation pairs contain a factor of the form $\varphi_{even}^{-1}(\zeta)$. Since all such factors contain ζ as a factor, then $\varphi_{even}^{-1}(\zeta = 0) \rightarrow \infty$ and the equations are uninterpretable. We have not as yet found a fundamental explanation of this nor a convenient way out of the dilemma. However, it is quite certain that the curve defined by these equations is a finite curve, for example any selected point on the B_o vs B_1 curve for $\zeta = 0$ is also one finite point on a B_o vs B_2 curve for which B_1 has the designated value. Further evidence of this was obtained as follows: equations for negative ζ were derived and B_o vs B_2 Mitrovic curves were calculated by digital computer for a third order polynomial using $\zeta = \mp .003$ and $\mp .001$. The results are shown on Fig. 2-1. The grouping of the curves is clear evidence that the B_o vs B_2 curve for $\zeta = 0$ must lie between the $\zeta = \mp .001$ curves, and thus must be finite. In practice use of a curve for $\zeta = 0.1$ should normally be satisfactory for engineering purposes.

In like manner for certain other pairs such as $B_o - B_3$ and $B_1 - B_4$ a factor of $\varphi_3^{-1}(\zeta)$ is obtained, which approaches infinity at $\zeta = 0.5$. If other similar singularities exist they have not yet been observed.

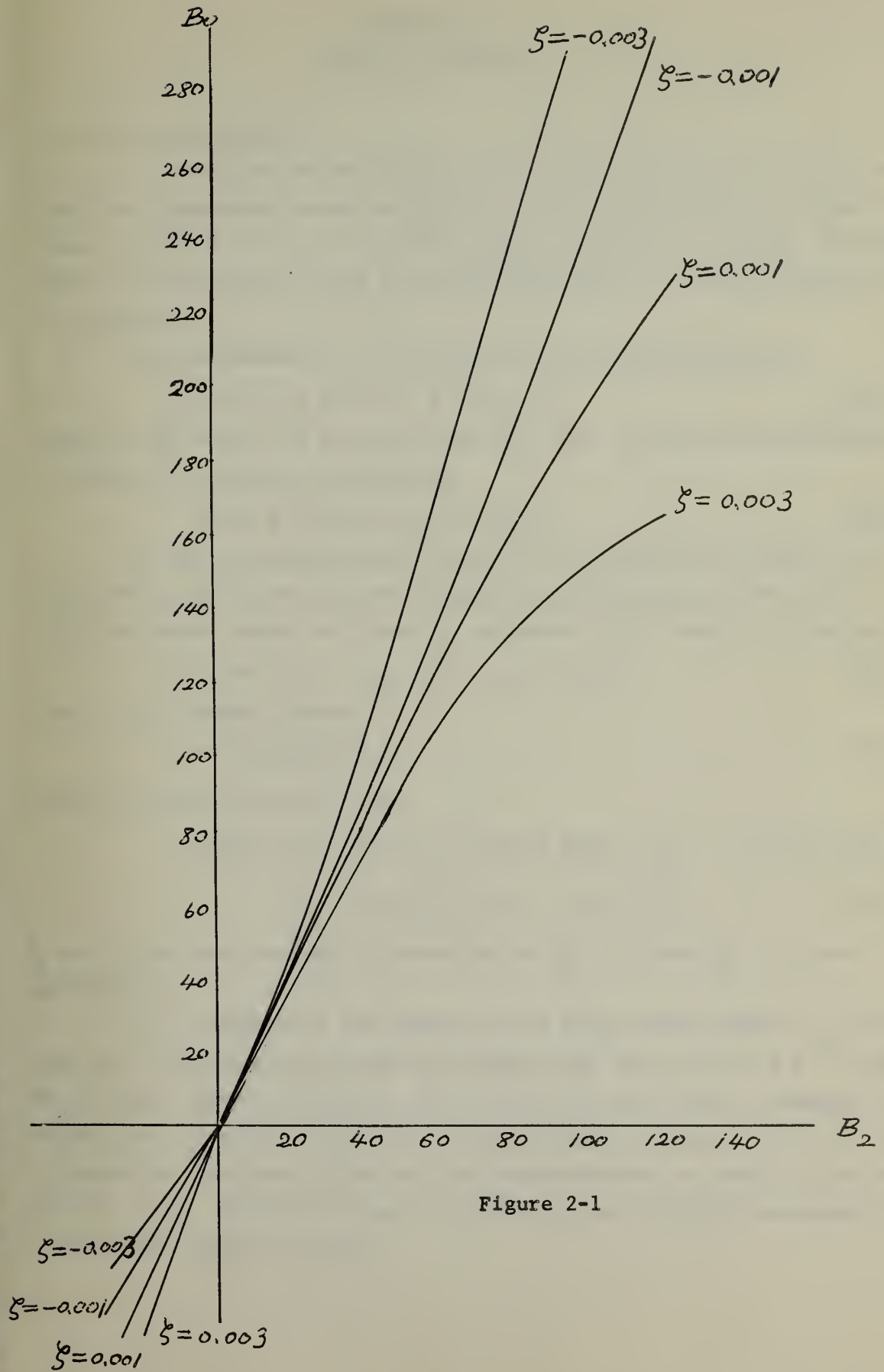

B_0
280
260
240
220
200
180
160
140
120
100
80
60
40
20
$\zeta = -0.003$
$\zeta = -0.001$
$\zeta = 0.001$
$\zeta = 0.003$
20
40
60
80
100
120
140
B_2
$\zeta = -0.003$
$\zeta = -0.001$
$\zeta = 0.003$
$\zeta = 0.001$

Figure 2-1

CHAPTER III

STABILITY CRITERIA

3.1 Basic Derivations

A stability criterion based only on the curve of $B_m - B_n$ on the real two dimensional plane is derived. Any interpretation in terms of a phase plot of F (s) on the complex plane is undesired, because the former curve is plotted first, and it is more convenient to skip the phase plotting, if possible.

Any polynomial F (s) can be put in the following form:

$$F(s) = f(s) + A_i s^i + A_j s^j \qquad (3\text{-}1)$$

which is not zero if s is not a root of F (s). Also the same polynomial in terms of B_i and B_j is given by:

$$F(s) = f(s) + B_i s^i + B_j s^j \qquad (3\text{-}2)$$

It must be noticed that this F (s) is zero for any value of s, provided B_i and B_j satisfy the relations given already in the Table. Thus for these values of B_i and B_j, equations (3-1) and (3-2) combine to:

$$F(s) = (A_i - B_i) s^i + (A_j - B_j) s^j \qquad (3\text{-}3)$$

Here s is a complex number

$$s = \omega_n \angle \pi/2 + \theta \qquad (3\text{-}4)$$

Substitute this into (3-3), and

$$\vec{F}(\omega_n) = (A_i - B_i)\,\omega_n^i \angle i(\pi/2 + \theta) + (A_j - B_j)\,\omega_n^j \angle j(\pi/2 + \theta)$$

$$= (A_i - B_i)\,\omega_n^i \vec{e}_i + (A_j - B_j)\,\omega_n^j \vec{e}_j. \qquad (3\text{-}5)$$

$\vec{e}_i$ and $\vec{e}_j$ are unit vectors in direction of $\frac{i\pi}{2} + i\theta$ and $\frac{j\pi}{2} + j\theta$ respectively.

It is evident if the phase plot of $\vec{F}(\omega_n)$ with regard to ω_n varying from zero to infinity encircles the origin just the angle of $n\pi/2 + n\theta$ in the counterclockwise direction, that all of the roots have a damping ratio ζ larger than $\zeta = \sin\theta$. The vector relationship in equation (3-5)can be used to determine some basic rules for the interpretation of stability on the Mitrovic plot. These rules are developed in the following paragraphs for most of the cases listed in Table I.

It should be noted that stability is determined by mapping the imaginary axis of the s-plane, ie., by the Mitrovic plot for $\zeta = 0$. A rule for this case is readily developed. When designing for good dynamic performance the location of the M - point can also determine whether a specification $\zeta \geq \zeta$, is satisfied. Rules for this are also developed.

(1) $B_o - B_1$

The criterion for this case has been given by Mitrovic for both $\zeta = 0$ and $\zeta > 0$ cases.

It must be noticed that this belongs to the special group of cases where the rule that B_o = const. must be cut first is valid even for $\zeta > 0$.

Rule:

For all of the roots to have a damping ratio ζ larger than $\sin\theta$, the B_o = const. line must be cut first. Cutting points must lie alternately on B_o = const. and B_1 = const. lines, as ω_n is increased from zero to infinity.

(2) $B_1 - B_2$

(i) $\zeta = 0$

This leads, θ being zero, to

$$\vec{e}_1 = \angle \pi/2, \quad \vec{e}_2 = \angle \pi,$$

from equation (3-5).

As is shown in Appendix II, it is valid in any pair that

$$\lim_{\omega_n \to o} \vec{F}(\omega_n) = A_o \vec{e}_o \qquad (3\text{-}6)$$

So the phase plot of $\vec{F}(\omega_n)$ starts from A_o on the real axis at $\omega_n = 0$, and must cut $\vec{e}_2$ first. This implies

$$A_2 = B_2 \text{ and } A_1 > B_1 \qquad (3\text{-}7)$$

at the first cutting, as can be seen from the equation:

$$\vec{F}(a_n) = (A_2 - B_2)\omega_n^2 \vec{e}_2 + (A_1 - B_1)\omega_n \vec{e}_1. \qquad (3\text{-}8)$$

This is illustrated in fig. 3-2; the provision (3-7) is realized when B_2 = const. line is cut first.

(ii) $\zeta > 0$

In this case each unit vector becomes

$$\vec{e}_1 = \angle \pi/2 + \theta, \quad \vec{e}_2 = \angle \pi + 2\theta.$$

As is seen from fig. 3-3 of the phase plot, $-\vec{e}_2$ must be cut first. This implies that

$$A_2 < B_2, \quad A_1 = B_1 \tag{3-9}$$

This is satisfied only when B_1 = const. line is cut first as shown in the same figure.

(3) $B_2 - B_3$

(i) $\zeta = 0$

For this case, from equation (3-5):

$$\vec{e}_2 = \angle \pi, \quad \vec{e}_3 = \angle 3\pi/2 .$$

Therefore the B_2 = const. line must be cut first as is explained in fig. 3-4.

(ii) $\zeta > 0$

Again applying equation (3-5), where

$$\vec{e}_2 = \angle \pi + 2\theta, \quad \vec{e}_3 = \angle 3\pi/2 + 3\theta,$$

it is easily seen the point cut first lies on the direction vector $-\vec{e}_2$, which leads to

$$A_3 = B_3, \quad A_2 < B_2.$$

Thus it is concluded that B_3, namely Bodd must be cut first in this case also, as shown in fig. 3-5.

(4) $B_3 - B_4$

(i) $\zeta = 0$

By the same analysis as the previous ones it is evident that the B_{even} = const. line must be cut first in order to have a stable domain.

(ii) $\zeta > 0$

The Bodd = const. line must be cut first.

(5) $B_o - B_3$

For this case equation (3-5) becomes

$$\vec{F}(\omega_n) = (A_3 - B_3)\,\omega_n^3\,\vec{e}_3 + (A_o - B_o)\,\vec{e}_o .$$

(i) $\zeta = 0$

$\vec{e}_3 = \angle 3\pi/2$, $\vec{e}_o = \angle 0$.

The phase plot shows $-\vec{e}_3$ must be cut first, which leads to

$A_3 < B_3$, $A_o = B_o$.

This implies, as is seen from fig. 3-6, that B_o const. line must be cut first in $B_3 - B_o$ plane.

(ii) $\zeta > 0$

$\vec{e}_3 = \angle 3\pi/2 + 3\theta$, $\vec{e}_o = \angle 0$.

The phase plot for this case shows that the point cut first must satisfy

$A_3 < B_3$, $A_o = B_o$,

which indicates, in terms of $B_o - B_3$ curve, that the B_o = const. line must be cut first as shown in fig. 3-7.

(6) $B_1 - B_4$

Equation (3-5) becomes

$\vec{F}(\omega_n) = (A_4 - B_4)\,\omega_n^4\,\vec{e}_4 + (A_1 - B_1)\,\omega_n\,\vec{e}_1$.

(i) $\zeta = 0$

$\vec{e}_4 = \angle 2\pi = \angle 0$, $\vec{e}_1 = \angle \pi/2$.

From the same analysis, it is evident that the $\vec{e}_1$ line must be cut first, which means, in terms of $B_1 - B_4$ plot, B_4 = const. line must be cut first.

(ii) $\zeta > 0$

$\vec{e}_4 = \angle 2\pi + 4\theta$, $\vec{e}_1 = \angle \pi/2 + \theta$

which indicates that the B_1 = const. line must be cut first.

All of the remaining cases in the Table have singularities at $\zeta = 0$, but this difficulty can be avoided by merely starting the value of ζ from a very small quantity. So $\zeta = 0$ case is dropped here intentionally.

(7) $B_o - B_2$

ζ is always assumed to be larger than zero, and unit vectors in equation (3-5) become:

$\vec{e}_2 = \angle \pi + 2\theta$, $\vec{e}_o = \angle 0$.

Fig. 3-8 shows that $-\vec{e}_2$ must be cut first, which means, in $B_o - B_2$ terminology, B_o = const. line must be cut first in order to make all of the roots have ζ larger than $\sin\theta$.

(8) $B_o - B_4$

In this case unit vectors of equation (3-5) are

$$\vec{e}_4 = \angle 4\theta \,, \quad \vec{e}_o = \angle 0 \,.$$

Almost the same analysis as before leads to the same conclusion: B_o = const. line must be cut first for our purpose.

(9) $B_1 - B_3$

$$\vec{e}_3 = \angle\left(\frac{3}{2}\pi + 3\theta\right) \,, \quad \vec{e}_1 = \angle\left(\frac{\pi}{2} + \theta\right)$$

Fig. 3-9 shows $\vec{e}_1$ must be cut first. This is interpreted, in $B_1 - B_3$ terminology, as B_3 = const. line must be cut first for the required purpose.

(10) $B_2 - B_4$

The same analysis requires that the B_4 = const. line should be cut first as the equation (3-5) becomes

$$\vec{F}(\omega_n) = (A_4 - B_4)\,\vec{e}_4 + (A_2 - B_2)\,\vec{e}_2$$

in this case.

Seeing these rules, it is desirable to classify them into a few groups to simplify them, as given in the following.

3.2 Summary of Enclosure Criteria - line cut first and the least number of cutting - points

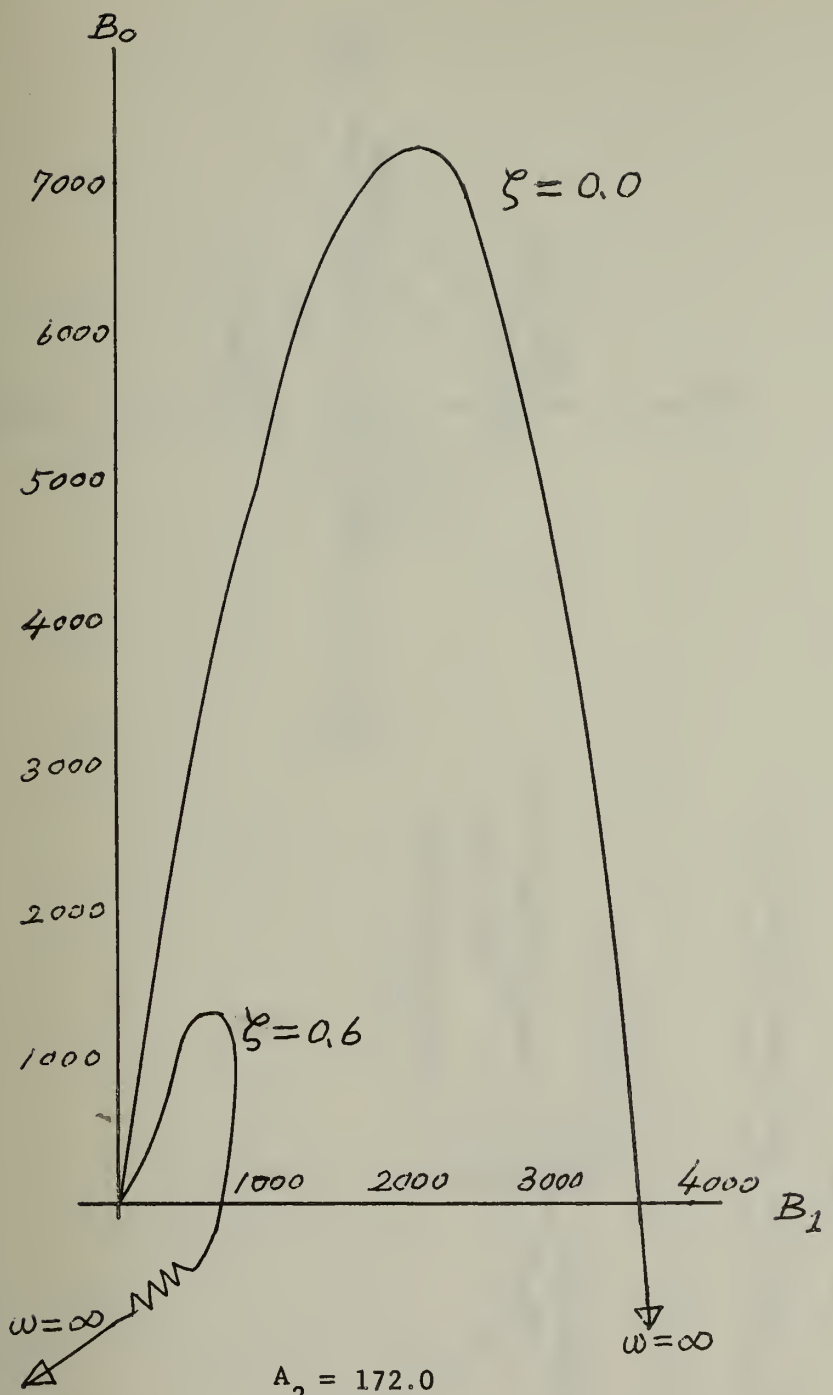
Case I	the line cut first $\zeta = 0$	least number of cutting points	the line cut first $\zeta > 0$	least number of cutting points
$B_o - B_1$	B_o	N - 1	B_o	N
$B_o - B_3$	B_o	N - 1	B_o	N
$B_1 - B_2$	B_2	N - 1	B_1	N + 1
$B_2 - B_3$	B_2	N - 1	B_3	N + 1
$B_3 - B_4$	B_4	N - 1	B_3	N + 1
$B_4 - B_1$	B_4	N - 1	B_1	N + 1

These six pairs permit the stability check in its strict sense, because they have no singularities at $\zeta = 0$, as has been already explained.

The next group consists of four pairs, all even - even and odd-odd and doesn't have the stability check in its narrowest sense, because all of them have singularities at $\zeta = 0$.

Case II	the line cut first $\zeta > 0$	least number of cutting points
$B_o - B_2$	B_o	N + 1
$B_o - B_4$	B_o	N + 1
$B_1 - B_3$	B_3	N + 1
$B_2 - B_4$	B_4	N + 1

In conclusion it can be said that the line for B_{even} = const. must be cut first when stability is checked in any case.

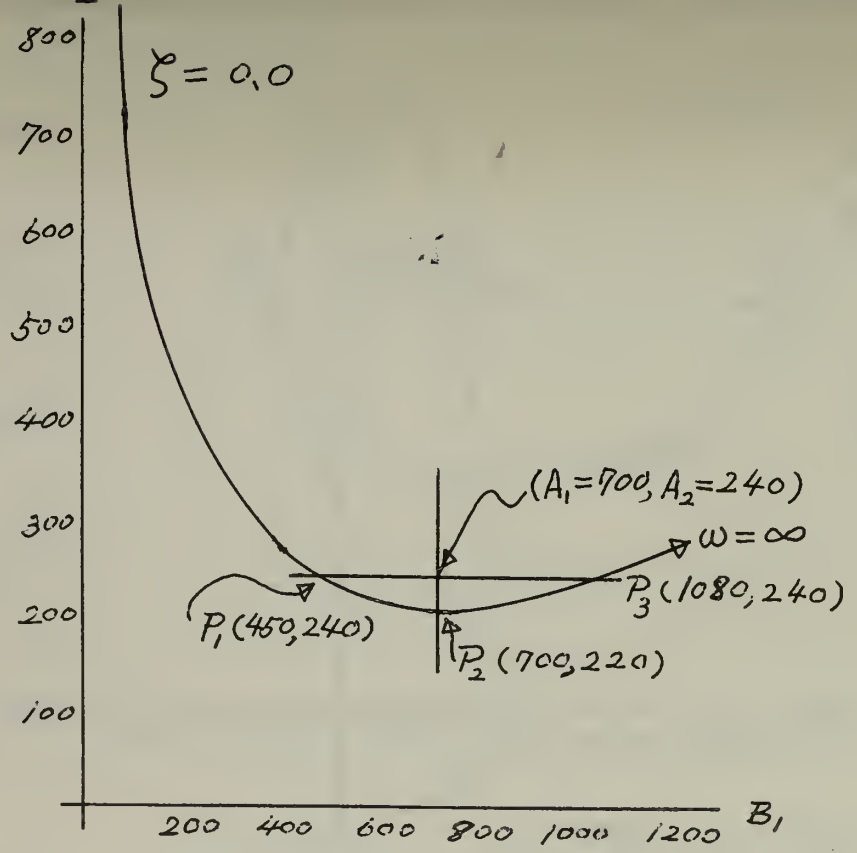


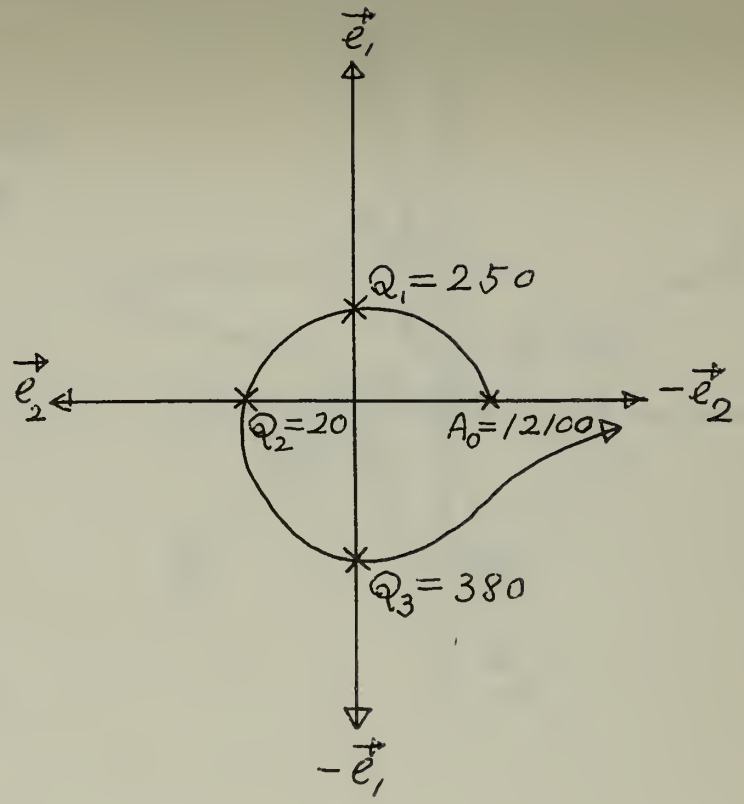
$A_2 = 172.0$

$A_3 = 22.0$

$A_4 = 1.0$

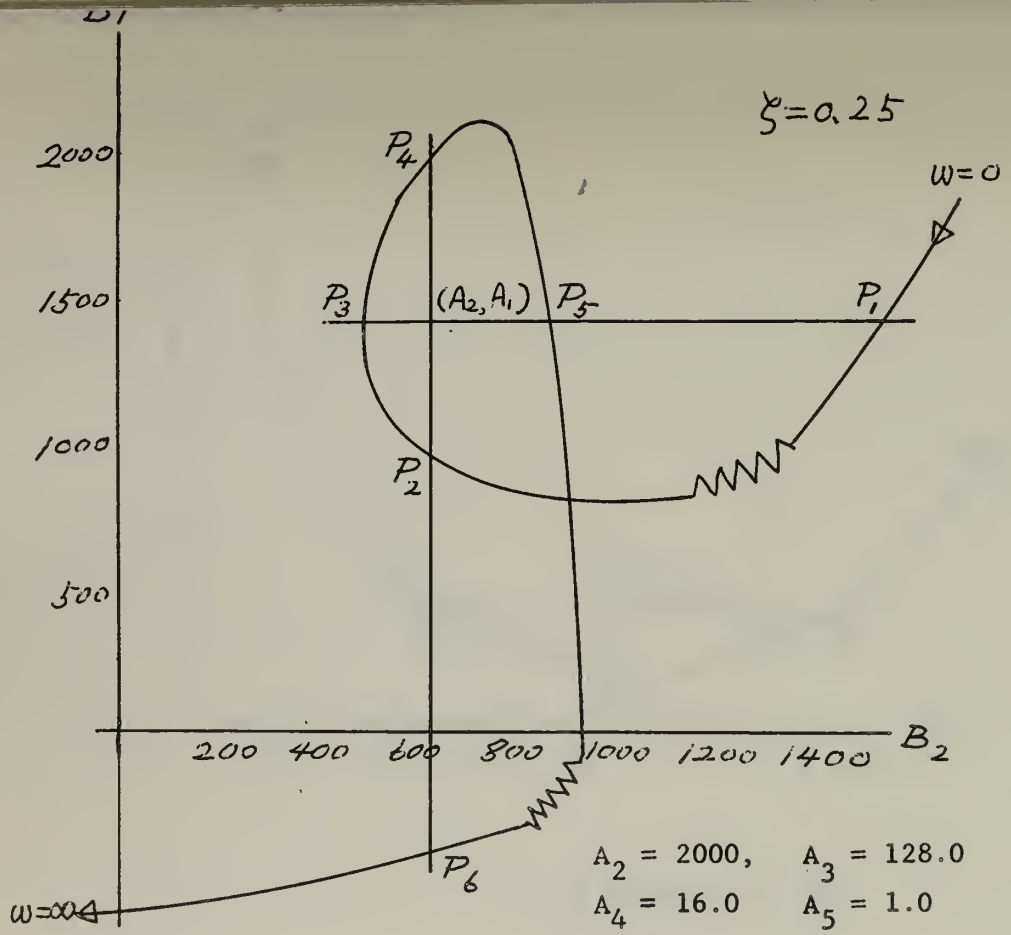
Figure 3-1





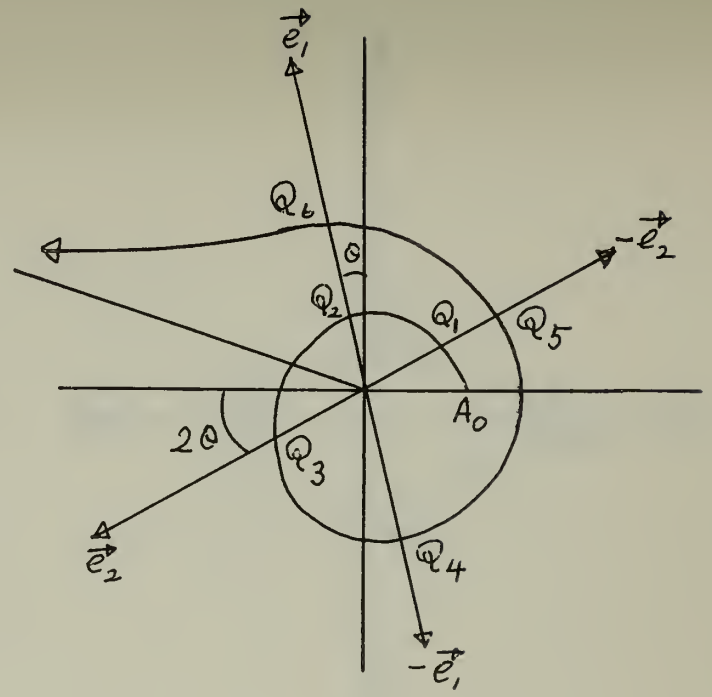
A_o = 12100
A_1 = 6.4
A_4 = 1.)

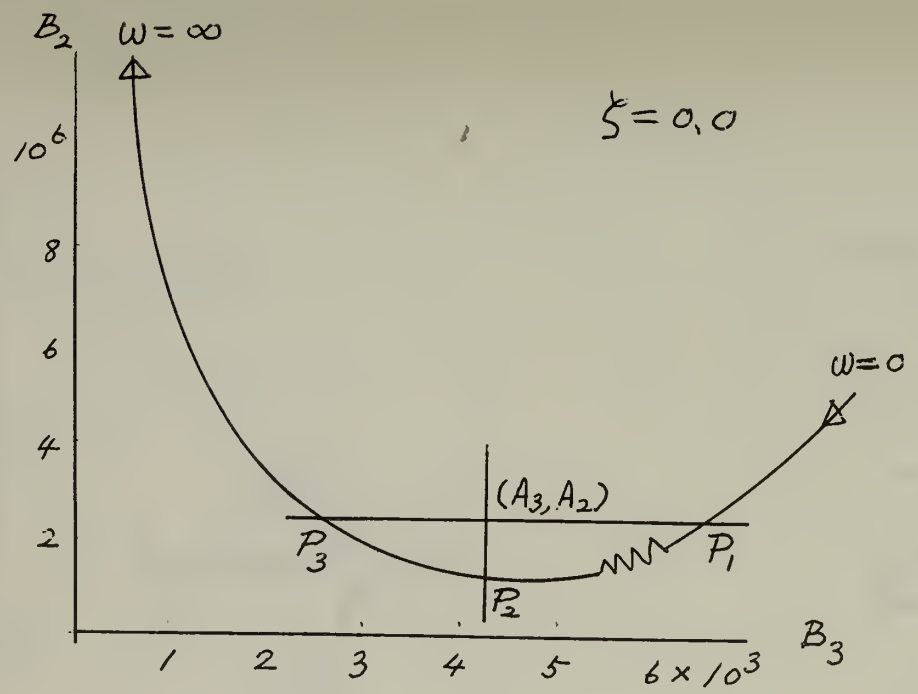
Figure 3-2

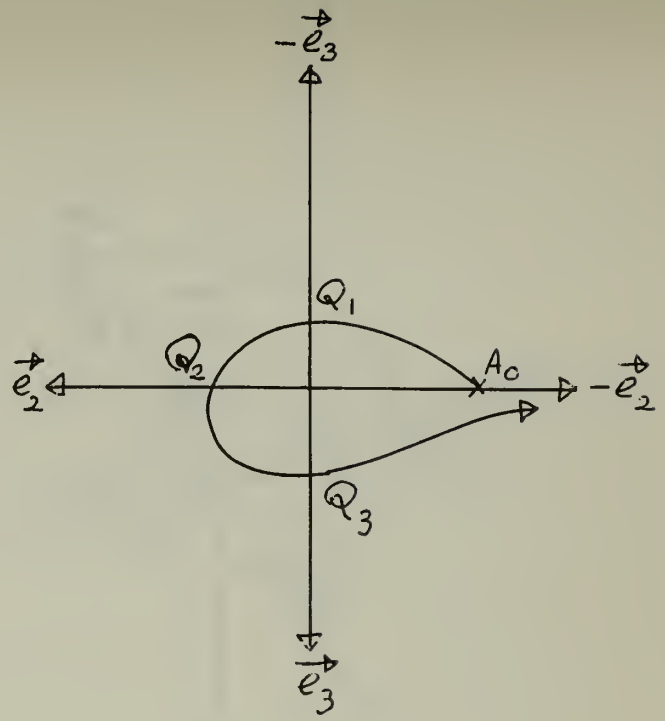


$A_2 = 2000, \quad A_3 = 128.0$

$A_4 = 16.0 \quad A_5 = 1.0$



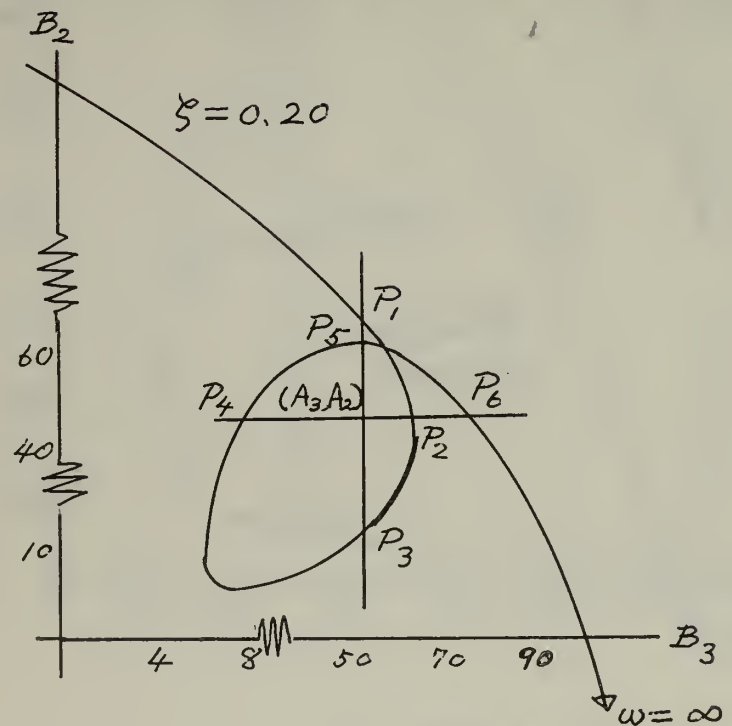


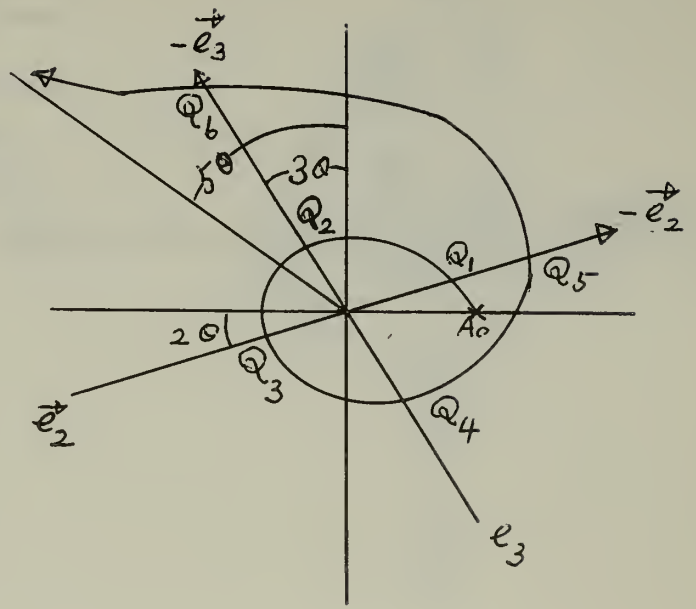


$A_o = 12100$, $A_1 = 682$

$A_4 = 1.0$

Figure 3-4

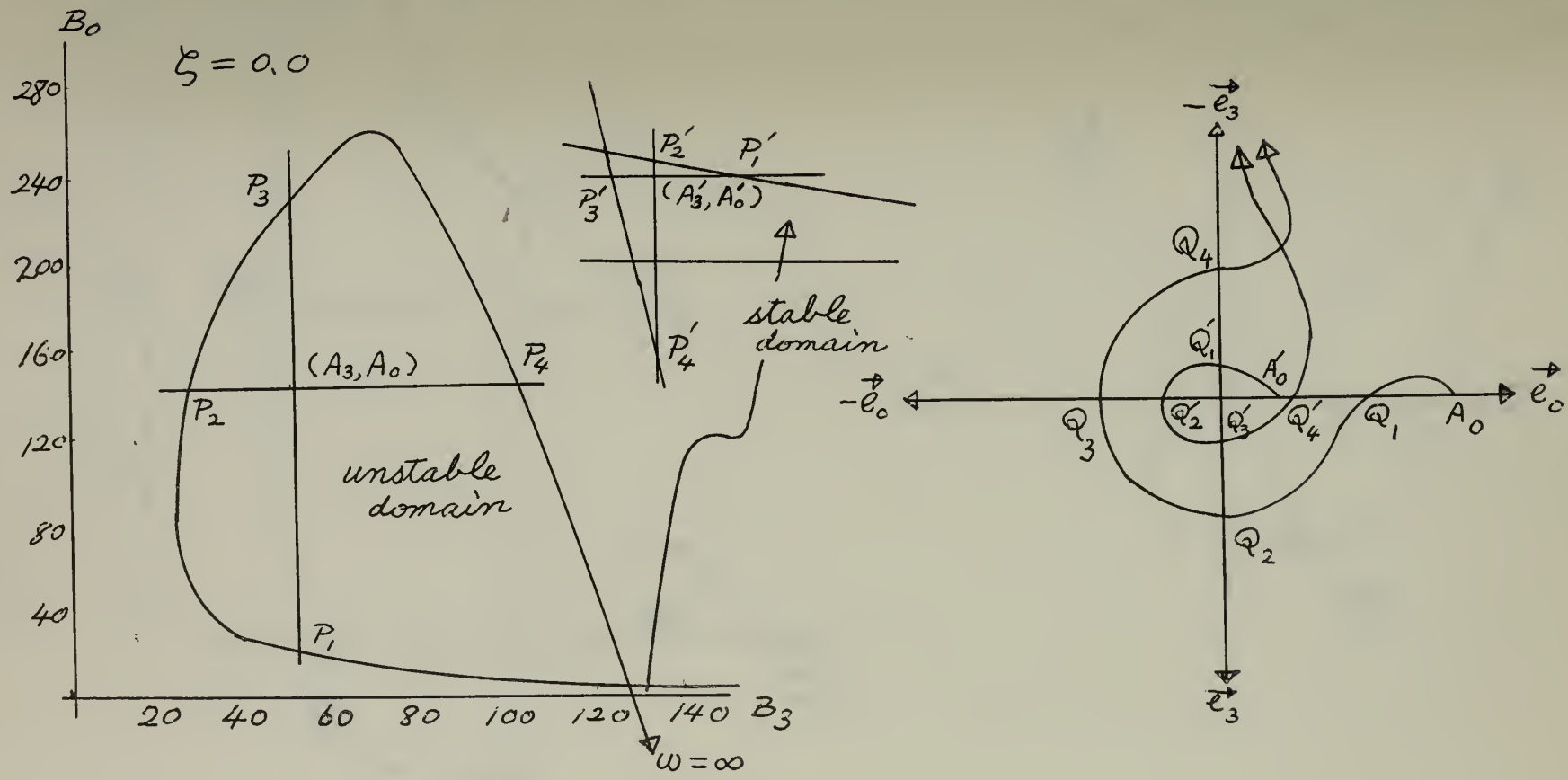




$A_o = 1, \quad A_1 = 4.1$

$A_4 = 41 \quad A_5 = 1$

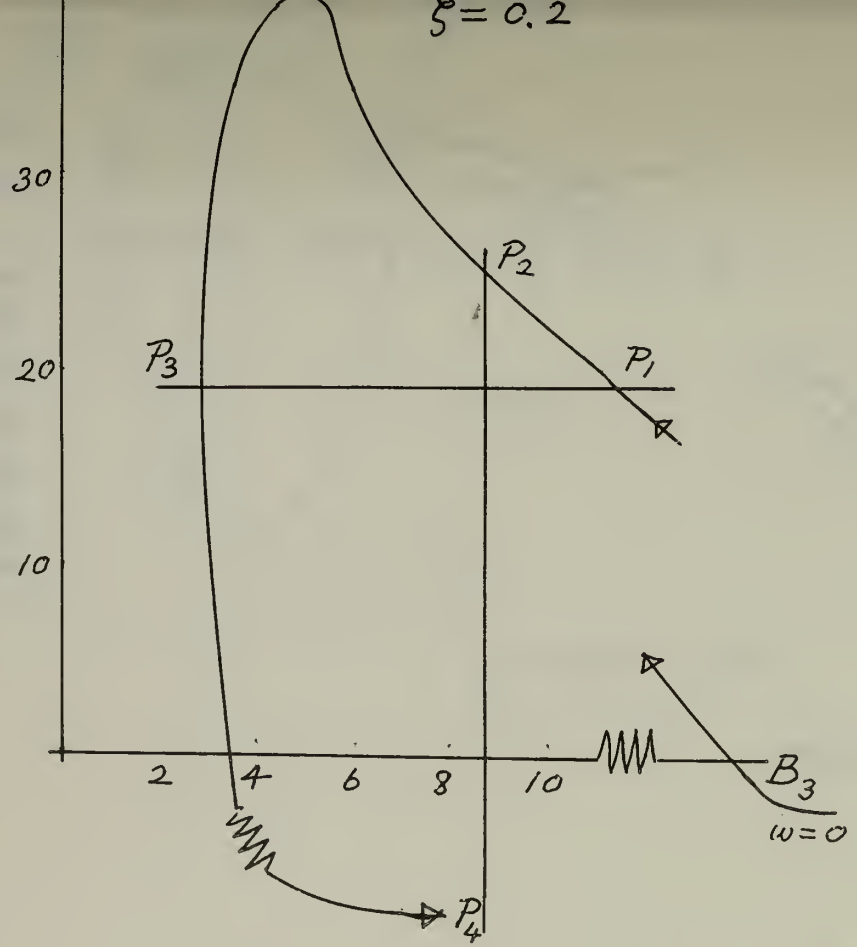
Figure 3-5

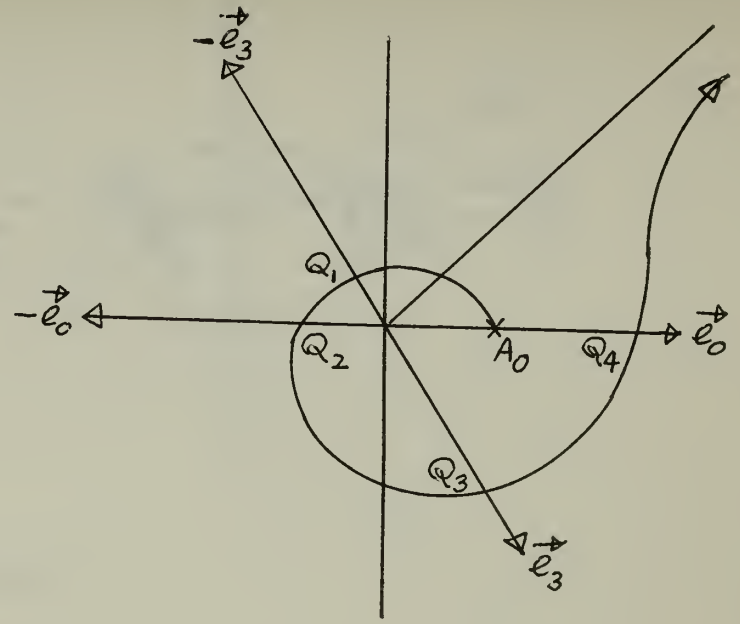


$A_1 = 2000$, $A_5 = 128.0$

$A_4 = 10$, $A_5 = 16$

Figure 3-6

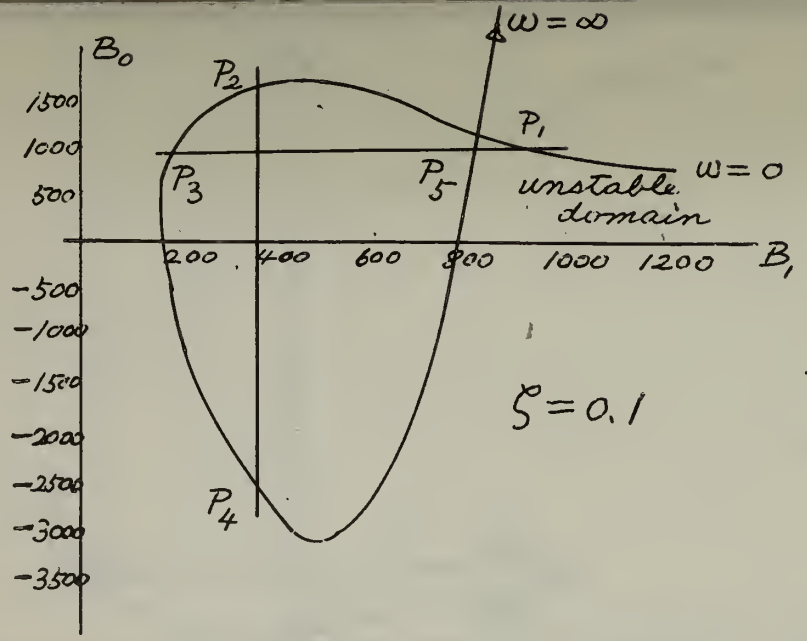


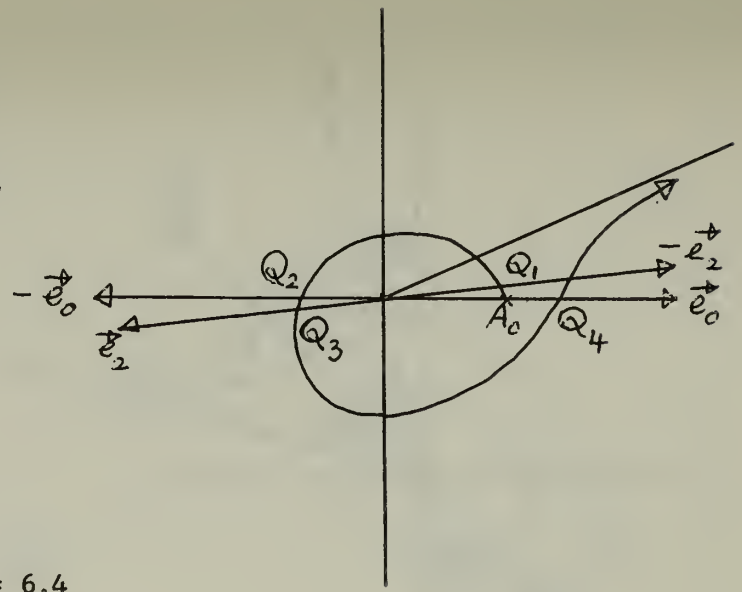


$A_1 = 30.6$, $A_2 = 16.4$

$A_4 = 1.0$

Figure 3-7

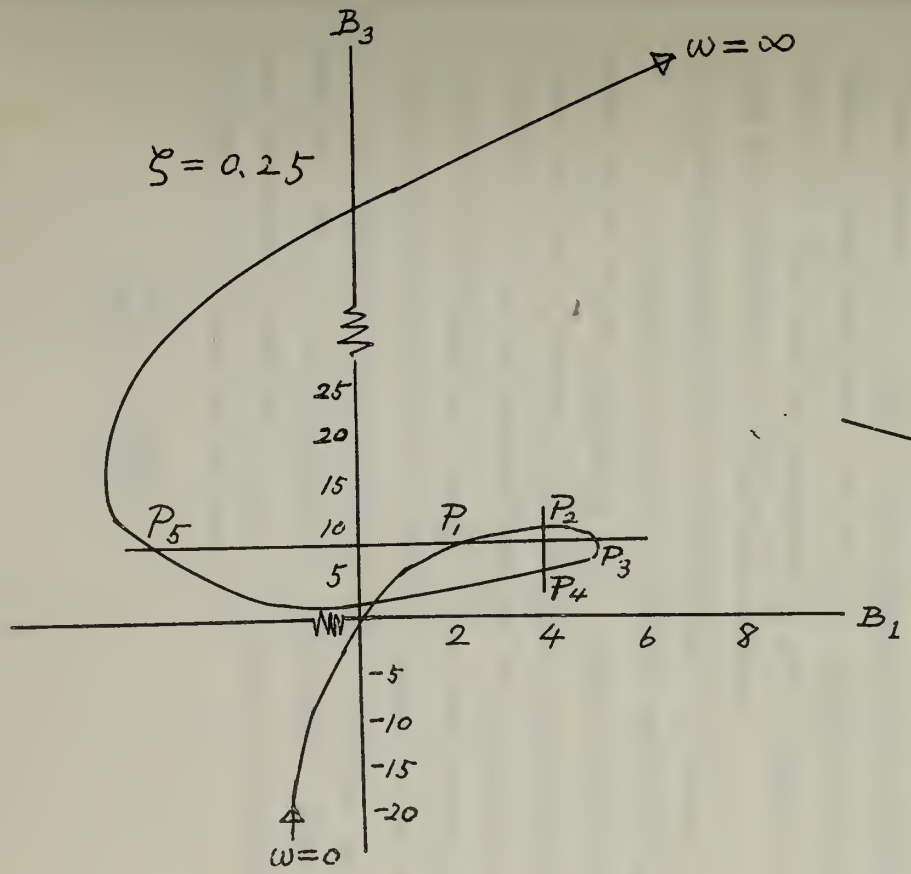


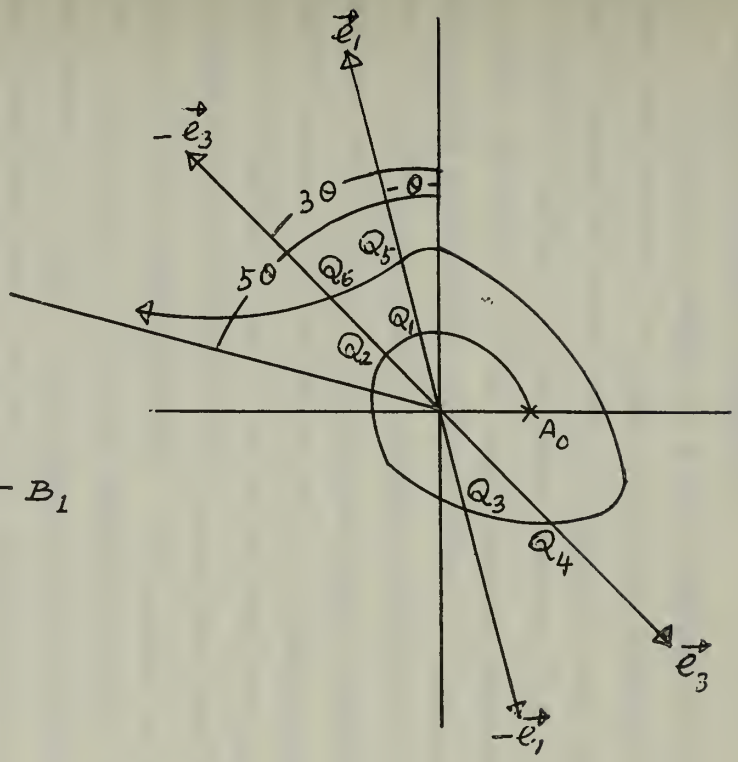


$A_1 = 682$, $A_3 = 6.4$

$A_4 = 1.0$

Figure 3-8





$A_o = 1.0$, $A_2 = 6.6$, $A_4 = 4.1$
$A_1 = 1.0$

Figure 3-9

CHAPTER IV

SKETCHING PROCEDURES

4.1 Need for sketching procedures.

Mitrovic's method is a very convenient method as far as calculations are concerned, because it involves the use of real numbers only. Mitrovic visualized use of desh calculators with secretarial type help, or a digital computer, or an analog computer. All of these techniques take time, and while they are acceptable for detailed studies, the engineer making preliminary studies prefers a technique (such as root locus or Bode diagram) which provides reasonably accurate curves with essentially no numerical calculations beyond formulation of the equations. Mitrovic's method will seldom be used for simple problems unless such a facility is available, and if it is not used for simple problems it is not likely that the engineer will develop the familiarity needed to apply it to more complex problems. Fortunately for most simple cases the Mitrovic curves for $\zeta = 0$ can be sketched rapidly, and for $\zeta \neq 0$ only a little labor is involved. Sketching was first used in the Elliott, Thaler, Heseltine paper, but was not developed as a technique for general use. The results presented in this chapter are intended to extend the usefulness of sketching methods.

4.2 Basic Manipulations in Sketching The Mitrovic Curves.

A Mitrovic curve is a plot of two equations with a common parameter which is the frequency, ω_n . Since the two equations are known, and since the plot (for sketching purposes) covers the range $0 \leq \omega_n \leq +\infty$ the locations of the two extreme ends of the curve are easily found by letting $\omega_n = 0$ in both equations and $\omega_n = \infty$ in both equations. With the two ends located evaluation of a few selected points on the curve permits sketching, and fortunately it is possible to locate maximum and minimum points which indicate the extreme excursions of the curve. Since the equations for the two adjustable coefficients B_x , B_y (also the coordinates of the plot!) are known it is easy to evaluate

$$\frac{dB_x}{d\omega_n} = 0; \quad \frac{dB_y}{d\omega_n} = 0$$

These equations can be evaluated for any order equations and for any value of ζ, but for sketching purposes $\zeta = 0$ is usually wanted since it clearly indicates the range of values that provide a stable system. Conveniently the Mitrovic equations have their simplest algebraic form when $\zeta = 0$ so that numerical formulation of the equation is quite elementary. Next the equations must be solved for their roots in order to find the values of ω_n at which the maxima and minima occur. For characteristic equations up to 5th order (and in some cases to 6th order) the polynomials are factorable and the results can be tabulated. This has been done for a reasonable number of cases and the results are tabulated in Table 4-1. Thus values of ω_n for the maxima and minima for any suitable* combination of coefficients, with $\zeta = 0$, can be found by referring to Table 4-1 and inserting the known values of the coefficients in the indicated relationships. It is then necessary to return to the original Mitrovic equations and substitute each of the values of ω_n, thus obtaining the coordinates of the maxima and minima points. The Mitrovic curve for $\zeta = 0$ may then be sketched by starting at $\omega_n = 0$ and connecting the known points in order of increasing ω_n, noting that each known point is also known to be either a maximum or a minimum. This procedure usually provides adequate information for preliminary analysis and design evaluation. Where a slightly more accurate curve is desired additional points may be calculated in the normal fashion, with the advantage that the known values of ω_n at the maxima and minima guide the choice of additional ω_n values.

When curves are desired for $\zeta \neq 0$, the same sketching procedures apply; and Table 4-1 also contains a number of entries for evaluation of maxima and minima. Note that when $\zeta \neq 0$ the Mitrovic equations are more complex algebraically so explicit solutions for the maxima and minima are restricted to equations of somewhat lower order.

For higher order equations, while sketching is still desirable, explicit solutions for the maxima and minima are not available. However, the general Mitrovic equations can always be differentiated and the differentiated forms are given here in Table 4-2. These can always be used to find the maxima and minima. Perhaps a simple graphical solution is fastest and most convenient: the proper equations are selected from Table 4-2 and into each selected values of ω_n are inserted, obtaining values for $\frac{d\, B_{x,y}}{d\, \omega_n}$

Cordinates		$\omega = 0$	$\omega = \infty$					ω_{max} or ω_{min}			
BO VS Bl	Highest Order	all	3	4	5	6	7	3	4	5	6
	Value of ζ	0	0					0			
	BO	0	$+\infty$	$-\infty$	$-\infty$	$+\infty$	$+\infty$	0	$0, \sqrt{A_2/2A_4}$	$0, \sqrt{A_2/2A_4}$	$0, \left[\left\{A_4 \pm \sqrt{A_4^2 - 3A_2A6}\right\}/3A_6\right]^{\frac{1}{2}}$
	Bl	0	$+\infty$	$-\infty$	$-\infty$	$-\infty$	$+\infty$	0	0	$0, \sqrt{A_3/2A_5}$	$0, \sqrt{A_3/2A_5}$

*

Continued	ω_{max} or ω_{min}	
Highest Order	7	8
Value of ζ	0	
BO	$0, \left[\left\{A_4 \pm \sqrt{A_4^2 - 3A_2A_6}\right\}/3A_6\right]^{\frac{1}{2}}$	too complicated
Bl	$0, \left[\left\{A_5 \pm \sqrt{A_5^2 - 3A_3A_7}\right\}/3A_7\right]^{\frac{1}{2}}$	too complicated

TABLE I (continued)

Coordinates		$\omega = 0$	$\omega = \infty$				ω_{max} or ω_{min}			
B1 VS B2	Highest Order	all	3	4	5	6	3	4	5	6
	Value of ζ	0	0				0			
	B1	0	$+\infty$	$+\infty$	$-\infty$	$-\infty$	0	0	$0, \sqrt{A_3/2A_5}$	$0, \sqrt{A_3/2A_5}$
	B2	$+\infty$	0	$+\infty$	$+\infty$	$-\infty$	∞	$(A_o/A_4)^{v4}$	$(A_o/A^4)^{\frac{1}{4}}$	too complicated

Coordinates		$\omega = 0$	$\omega = \infty$				ω_{max} or ω_{min}			
B2 VS B3	Highest Order	all	3	4	5	6	3	4	5	6
	Value of ζ	0	0				0			
	B2	$+\infty$	0	$+\infty$	$+\infty$	$-\infty$	$+\infty$	$(A_o/A_4)^{\frac{1}{4}}$	$(A_o/A_4)^{\frac{1}{4}}$	too complicated
	B3	$+\infty$	0	0	$+\infty$	$+\infty$	$+\infty$	$+\infty$	$(A_1/A5)^{\frac{1}{4}}$	$(A_1/A_5)^{\frac{1}{4}}$

TABLE I (continued)

Coordinates		$\omega = 0$	$\omega = \infty$			ω_{max} or ω_{min}		
B3 VS B4	Highest Order	all	4	5	6	4	5	6
	Value of ζ	0	0			0		
	B 3	$+\infty$	0	$+\infty$	$+\infty$	∞	$(A_1/A_5)^{\frac{1}{4}}$	$(A_1/A_5)^{\frac{1}{4}}$
	B 4	$-\infty$	0	0	$+\infty$	$\sqrt{2\,A_o/A_2}, \infty$	$\sqrt{2\,A_o/A_2}, \infty$	too complicated

Coordinates		$\omega = 0$	$\omega = \infty$				ω_{max} or ω_{min}			
BO VS B3	Highest Order	all	3	4	5	6	3	4	5	6
	Value of ζ	0	0				0			
	B 0	0	$+\infty$	$-\infty$	$-\infty$	$+\infty$	0	$0, \sqrt{A_2/2\,A_4}$	$0, \sqrt{A_2/2\,A_4}$	too complicated
	B 3	$+\infty$	0	0	$+\infty$	$+\infty$	∞	∞	$(A_1/A_5)^{\frac{1}{4}}$	$(A_1/A_5)^{\frac{1}{4}}$

Coordinates		$\omega = 0$	$\omega = \infty$			ω_{max} or ω_{min}		
B1 VS B4	Highest Order	all	4	5	6	4	5	6
	Value of ζ	0	0			0		
	B 1	0	$+\infty$	$-\infty$	$-\infty$	0	$0, \sqrt{A_3/2A_5}$	$0, \sqrt{A_3/2A_5}$
	B 4	$-\infty$	0	0	$+\infty$	$\sqrt{2A_0/A_2}, \infty$	$\sqrt{2A_0/A_2}, \infty$	too complicated

*

Coordinates		$\omega = 0$	$\omega = \infty$					ω_{max} or ω_{min}
BO VS B1	Highest Order	all	3		4			3
	Value of ζ	$0 < \zeta \leq 1$	$0 < \zeta \leq 0.5$	$0.5 < \zeta \leq 1$	$0 < \zeta \leq 0.5$	$0.5 < \zeta < 0.71$	$0.71 \leq \zeta \leq 1$	
	B 0	0	$-\infty$	$-\infty$	$-\infty$	$+\infty$	$+\infty$	$0, 2A_2/3A_3\varphi_2(\zeta)$
	B 1	0	$+\infty$	$-\infty$	$-\infty$	$-\infty$	$+\infty$	$-A_2\varphi_2(\zeta)/A_3\varphi_3(\zeta)$

Continued	ω_{max} or ω_{min}
Highest Order	4
Values of ζ	
B 0	$0, \left\{-3A_3 \varphi_2(\zeta) \pm \sqrt{9A_3^2 \varphi_2^2(\zeta) + 32A_2A_4 \varphi_3(\zeta)}\right\} / 8A_4 \varphi_3(\zeta)$
B 1	$\left\{-A_3 \varphi_3(\zeta) \pm \sqrt{A_3^2 \varphi_3^2(\zeta) - 3A_2A_4 \varphi_2(\zeta) \varphi_4(\zeta)}\right\} 3A_4 \varphi_4(\zeta)$

Coordinates		$\omega = 0$	$\omega = \infty$			ω_{max} or ω_{min}	
B1 VS B2	Highest Order	all	3	4		3	4
	Values of ζ	$0 < \zeta \leq 1$	$0 < \zeta \leq 1$	$0 < \zeta \leq 0.5$	$0.5 < \zeta \leq 1$		
	B 1	$+\infty$	$+\infty$	$-\infty$	$-\infty$	$(A_o \varphi_2(\zeta)/2A_3)^{1/3}$	too complicated
	B 2	$+\infty$	$+\infty$	$+\infty$	$-\infty$	$(2A_o/A_3 \varphi_2(\zeta)^{1/3}$	too complicated

Coordinates		$\omega = 0$		$\omega = \infty$		ω_{max} or ω_{min}	
B2 VS B3	Highest Order	all		3	4	3	4
	Values of ζ	$0 < \zeta \leq 0.5$	$0.5 < \zeta \leq 1$	$0 < \zeta \leq 1$	$0 < \zeta \leq 1$		
	B 2	$+\infty$	$-\infty$	0	$+\infty$	$-2A_o \varphi_3(\zeta)/A_1\varphi_2(\zeta)$, ∞	too complicated
	B 3	$-\infty$	$-\infty$	0	$+\infty$	$3A_o \varphi_2(\zeta)/2A_1$, ∞	too complicated

*

Coordinates		$\omega = 0$			$\omega = \infty$	
B3 VS B4	Highest Order	all			4	5
	Values of ζ	$0 < \zeta \leq 0.5$	$0.5 \leq \zeta < 0.71$	$0.71 \leq \zeta < 1$	$0 < \zeta \leq 1$	$0 < \zeta \leq 1$
	B 3	$-\infty$	$-\infty$	$+\infty$	0	$+\infty$
	B 4	$-\infty$	$+\infty$	$+\infty$	0	$+\infty$

Continued	ω_{max} or ω_{min}	
Highest Order	4	5
Values of ζ		
B 3	$\left\{-2A_1 \varphi_3(\zeta) \pm \sqrt{A_1^2 \varphi_3^2(\zeta) - 3A_o A_2 \varphi_2(\zeta) \varphi_3(\zeta)}\right\} / A_2 \varphi_2(\zeta), \infty$	too complicated
B 4	$\left\{3A_1 \varphi_2(\zeta) \pm \sqrt{9A_1^2 \varphi_2^2(\zeta) + 32 A_o A_2 \varphi_3(\zeta)}\right\} / 4A_2, \infty$	too complicated

Coordinates		$\omega = 0$		ω_{max} or ω_{min}				
	Highest Order	all		3		4		
BO VS B3	Values of ζ	$0 < \zeta < 0.5$	$0.5 < \zeta \leq 1$	$0 < \zeta < 0.5$	$0.5 < \zeta \leq 1$	$0 < \zeta < 0.5$	$0.5 < \zeta < 0.71$	$0.71 \leq \zeta \leq 1$
	B 0	0	0	$+\infty$	$-\infty$	$-\infty$	$+\infty$	$+\infty$
	B 3	$+\infty$	$-\infty$	$-\infty$	$-\infty$	$+\infty$	$-\infty$	$+\infty$

Continued	ω_{max} or ω_{min}	
Highest Order	3	4
Values of ζ		
B 0	$A_1 \varphi_2(\zeta)/2 A_2$	too complicated
B 3	$2 A_1/A_2 \varphi_2(\zeta), \infty$	too complicated

Coordinates		$\omega = 0$			$\omega = \infty$		ω_{max} or ω_{min}
B1 VS B4	Highest Order	all			4		4
	Values of ζ	$0 < \zeta < 0.5$	$0.5 < \zeta < 0.71$	$0.71 < \zeta < 1$	$0 < \zeta < 0.5$	$0.5 < \zeta < 1$	
	B 1	$+\infty$	$-\infty$	$+\infty$	$+\infty$	$-\infty$	too complicated
	B 4	$-\infty$	$+\infty$	$+\infty$	0	0	too complicated

TABLE I (continued)

Coordinates		$\omega = 0$	$\omega = \infty$			
BO VS B2	Highest Order	all	3		4	
	Values of ζ	$0 < \zeta \leq 1$	$0 < \zeta < 0.5$	$0.5 \leq \zeta \leq 1$	$0 < \zeta \leq 0.71$	$0.71 < \zeta \leq 1$
	B O	0	$-\infty$	$-\infty$	$+\infty$	$+\infty$
	B 2	∞	$-\infty$	$+\infty$	$+\infty$	$-\infty$

*

Continued	ω_{max} or ω_{min}	
Highest Order	3	4
Value of ζ		
BO	$A_1 \varphi_2(\zeta)/2A_2$	too complicated
B2	$2 A_1/A_2 \varphi_2(\zeta), \infty$	too complicated

TABLE I (continued)

Coordinates		$\omega = 0$		$\omega = \infty$				
B1 VS B3	Highest Order	all		3	4		5	
	Values of ζ	$0 < \zeta < 0.5$	$0.5 \leq \zeta \leq 1$	$0 < \zeta \leq 1$	$0 < \zeta < 0.5$	$0.5 \leq \zeta \leq 1$	$0 < \zeta \leq 0.71$	$0.71 < \zeta \leq 1$
	B1	$-\infty$	$+\infty$	$+\infty$	$-\infty$	$-\infty$	$+\infty$	$+\infty$
	B3	$-\infty$	$-\infty$	0	$-\infty$	$+\infty$	$+\infty$	$-\infty$

Continued	ω_{max} or ω_{min}	ω_{max} or ω_{min}	
Highest Order	3	4	5
Values of ζ			
B 1	$\sqrt{A_o \varphi_3(\zeta)/A_2}$	$\left[\left\{A_2 \pm \sqrt{A_2^2 + 12\, A_o \varphi_3(\zeta)}\right\}/6A_4\right]^{1/2}$	too complicated
B 3	$\sqrt{3A_o/A_2}, \infty$	$\left[\left\{-A_2 \pm \sqrt{A_2^2 - 12A_oA_3 \varphi_3(\zeta)}\right\}/2A_4\right]^{1/2}$	too complicated

TABLE I (continued)

Coordinates		$\omega = 0$		$\omega = \infty$	ω_{max} or ω_{min}
B2 VS B4	Highest Order	all		4	4
	Values of ζ	$0 < \zeta \leq 0.71$	$0.71 < \zeta \leq$	$0 < \zeta \leq 1$	
	B2	$+\infty$	$-\infty$	$+\infty$	too complicated
	B4	$+\infty$	$+\infty$	0	

*

Coordinates		$\omega = 0$		$\omega = \infty$				
BO VS B4	Highest Order	all		4		5		
	Values of ζ	$0 < \zeta < 0.71$	$0.71 < \zeta \leq 1$	$0 < \zeta < 0.71$	$0.71 < \zeta \leq 1$	$0 < \zeta < 0.31$	$0.31 \leq \zeta < 0.71$	$0.71 < \zeta \leq 1$
	BO	0	0	$+\infty$	$-\infty$	$-\infty$	$-\infty$	$+\infty$
	B4	$-\infty$	$+\infty$	0	0	$-\infty$	$+\infty$	$+\infty$

* TABLE I (continued)

Continued	ω_{max} or ω_{min}	
Highest Order	4	5
Values of ζ		
BO	$\left\{A_o \varphi_2(\zeta) \pm \sqrt{A_2^2 \varphi_2^2(\zeta) + 3A_1 A_3(\zeta)}\right\}/3A_3$	too complicated
B4	$\left\{-A_2 \varphi_2(\zeta) \pm \sqrt{A_2^2(\zeta) + 3 A_1 A_3 \varphi_3(\zeta)}\right\} A_3 \varphi_3(\zeta)$	too complicated

at selected ω_n values. A plot is then made of $\frac{dB_x}{d\omega_n}$ vs ω_n and $\frac{dB_y}{d\omega_n}$ vs ω_n. Each curve crosses the ω_n axis at values of ω_n corresponding to the maxima and minima.

4.3 Tabulation of Results and Suggested Procedures.

(Tables containing useful relationships are collected in the pages following.

Suggested Procedures: (these are largely a summary of the discussion in section 4.2)

1. From the characteristic equation determine the desired Mitrovic equations for $\zeta = 0$.
2. Substitute in these equations $\omega_n = 0$ and $\omega_n = \infty$ to locate the ends of the curve.
3. Inspect the Mitrovic equations to see if a ready solution exists for zero points; ie., for what values of ω_n are $B_x = 0$, $B_y = 0$?
4. Obtain the equations $\frac{dB_x}{d\omega_n} = 0$, $\frac{dB_y}{d\omega_n} = 0$ and solve for values of ω_n.
5. Substitute the values obtained in 3 and 4 into the Mitrovic equations to locate the B_x, B_y coordinates.
6. Plot the known points and sketch the curve.

4.4 Illustrations of the Sketching Techniques

Example I

For a third order equation B_o vs B_1, $\zeta = 0$

$$F(s) = s^3 + 2s^2 + B_1 s + B_o$$

$\zeta = 0$

$$B_o = 2\omega^2$$
$$B_1 = \omega^2 \qquad (4\text{-}1)$$

From the relationships in the table, B_o and B_1 are both equal to zero at $\omega = 0$; they have no maximum or minimum points; both end in $+\infty$. Thus for equation 4-1, a sketch of B_o - B_1 curve is shown in Fig. 4-1. Notice: all of the roots of F(s) have a damping ratio $\zeta \geq 0.5$ if $B_o = 1$, $B_1 = 2$.

TABLE 4 - 2

DIFFERENTIATED FORMS OF MITROVIC' EQUATIONS

FOR USE IN EVALUATING MAXIMA AND MINIMA

BO VS B2

$$\frac{d\,B\,\theta}{d\,\omega} = -\left[-2\,A_2\omega + 3\,A_3\,\varphi_2\,(\zeta)\,\omega^2 + \ldots + n\,A_n\,\varphi_{n-1}(\zeta)\,\omega^{n-1}\right]$$

$$\frac{d\,B\,1}{d\,\omega} = A_2\,\varphi_2\,(\zeta) + 2\,A_3\,\varphi_3\,(\zeta)\,\omega + \ldots + (n-1)\,A_n\,\varphi_n\,(\zeta)\,\omega^{n-2}$$

BI VS B2

$$\frac{d\,B\,1}{d\,\omega} = -A_o\,\varphi_2\,(\zeta)\,\frac{1}{\omega_2} + 2\,A_3\,\omega - 3\,A_4\,\varphi_2\,(\zeta)\,\omega^2 - \ldots - (n-1)\,A_n\,\varphi_{n-2}\,(\zeta)\,\omega^{n-2}$$

$$\frac{d\,B\,2}{d\,\omega} = -2\,\frac{A_o}{\omega^3} + A_3\,\varphi_2\,(\zeta) + 2\,A_4\,\varphi_3\,(\zeta)\,\omega + \ldots + (n-2)\,A_n\,\varphi_{n-1}(\zeta)\,\omega^{n-3}$$

B2 VS B3

$$\frac{d\,B\,2}{d\,\omega} = -2\,A_o\,\varphi_3\,(\zeta)\,\frac{1}{\omega^3} - A_1\,\varphi_2\,(\zeta)\,\frac{1}{\omega^2} + 2\,A_4\,\omega + \ldots + (n-2)\,A_n\,\varphi_{n-3}\,(\zeta)\,\omega^{n-3}$$

$$\frac{d\,B\,3}{d\,\omega} = 3\,A_o\,\varphi_2\,(\zeta)\,\frac{1}{\omega^4} - 2\,A_1\,\frac{1}{\omega^3} + A_4\,\varphi_2\,(\zeta) + \ldots + (n-3)\,A_n\,\varphi_{n-2}\,\omega^{n-4}$$

B3 VS B4

$$\frac{d\,B\,3}{d\,w} = -3\,A_o\,\varphi_4(\zeta)\frac{1}{\omega^4} - 2A_1\,\varphi_3\,(\zeta)\frac{1}{\omega^3} - A_2'\,\varphi_2\,(\zeta)\frac{1}{\omega^2} + 2A_5\omega - \ldots - (n-3)\,A_n\,\varphi_{n-4}(\zeta)\,\omega^{n-4}$$

$$\frac{d\,B\,4}{d\,\omega} = 4\,A_o\,\varphi_3\,(\zeta)\frac{1}{\omega^5} + 3\,A_1\,\varphi_2\,(\zeta)\frac{1}{\omega^4} - 2\,A_2\frac{1}{\omega^3} + A_5\,\varphi_2\,(\zeta) + \ldots + (n-4)\,A_n\,\varphi_{n-3}\,(\zeta)\,\omega^{n-5}$$

TABLE 4-2 (Continued)

B0 VS B3

$$\frac{d\,B\,0}{d\,\omega} = \frac{1}{\varphi_3(\zeta)}\left[-A_1\,\varphi_2\,(\zeta) + 2\,A_2\omega - 4A_4\,\omega^3 + \ldots + nA_n\,\varphi_{n-3}\,(\zeta)\,\omega^{n-1}\right]$$

$$\frac{d\,B\,3}{d\,\omega} = \frac{1}{\varphi_3(\zeta)}\left[-\frac{2-A_1}{\omega^3} + A_2\,\varphi_2\,(\zeta)\frac{1}{\omega^2} - A_4\,\varphi_4\,(\zeta) - \ldots - (n-3)\,A_n\,\varphi_n\,(\zeta)\,\omega^{n-4}\right]$$

B1 VS B4

$$\frac{d\,B\,1}{d\,\omega} = \frac{1}{\varphi_3(\zeta)}\left[A_o\,\varphi_4\,(\zeta)\frac{1}{\omega^2} - A_2\,\varphi_2\,(\zeta) + 2\,A_3\,\omega + \ldots + (n-1)\,A_n\,\varphi_{n-4}\,(\zeta)\,\omega^{n-2}\right]$$

$$\frac{d\,B\,4}{d\,\omega} = \frac{1}{\varphi_3(\zeta)}\left[\frac{4A_o}{\omega^5} + 2\,\frac{A_2}{\omega^3} + A_3\,\varphi_2\,(\zeta)\frac{1}{\omega^2} - \ldots - (n-4)\,A_n\,\varphi_{n-1}\,(\zeta)\,\omega^{n-5}\right]$$

B0 VS B2

$$\frac{d\,B\,0}{d\,\omega} = \frac{1}{\varphi_2(\zeta)}\left[A_1 - 3\,A_1\,\omega^2 + 4\,A_4\,\varphi_2\,(\zeta)\,\omega^3 + \ldots + n\,A_n\,\varphi_{n-2}\,(\zeta)\,\omega^{n-1}\right]$$

$$\frac{d\,B\,2}{d\,\omega} = \frac{-1}{\varphi_2(\zeta)}\left[\frac{A_1}{\omega^2} + A_3\,\varphi_3\,(\zeta) + 2\,A_4\,\varphi_4\,(\zeta)\,\omega + \ldots + (n-2)\,A_n\,\varphi_n\,(\zeta)\,\omega^{n-3}\right]$$

B1 VS B3

$$\frac{d\,B\,1}{d\,\omega} = \frac{1}{\varphi_2\,(\zeta)}\left[A_o\,\varphi_3\,(\zeta)\frac{1}{\omega^2} + A_2 - 3\,A_4\,\omega^2 + 4\,A_5\,\varphi_2\,(\zeta)\,\omega^3 + \ldots + (n-1)\,A_n\,\varphi_{n-3}(\zeta)\,\omega^{n-2}\right]$$

$$\frac{d\,B\,3}{d\,\omega} = \frac{1}{\varphi_1(\zeta)}\left[\frac{4\,A_o}{\omega^4} - \frac{A_2}{\omega^2} - A_4\,\varphi_3\,(\zeta) - 2\,A_5\,\varphi_4\,(\zeta)\,\omega - \ldots - (n-3)\,A_n\,\varphi_{n-1}\,(\zeta)\,\omega^{n-4}\right]$$

B2 VS B4

$$\frac{d\,B\,2}{d\,\omega} = \frac{1}{\varphi_2(\zeta)} \left[2\,A_o\,\varphi_4\,(\zeta)\,\frac{1}{\omega^3} + A_1\,\varphi_3\,(\zeta)\,\frac{1}{\omega^2} + \ldots + (n-2)\,A_n\,\varphi_{n-4}\,(\zeta)\,\omega^{n-3} \right]$$

$$\frac{d\,B\,4}{d\,\omega} = \frac{1}{\varphi_2(\zeta)} \left[-4\,A_o\,\varphi_2\,(\zeta)\,\frac{1}{\omega^5} + \frac{3\,A_1}{\omega^4} - \frac{A_3}{\omega^2} - \ldots - (n-4)\,A_n\,\varphi_{n-2}\,(\zeta)\,\omega^{n-5} \right]$$

BO VS B4

$$\frac{d\,B\,0}{d\,\omega} = \frac{1}{\varphi_4(\zeta)} \left[A_1\,\varphi_3\,(\zeta) + 2\,A_2\,\varphi_2\,(\zeta)\,\omega - 3\,A_3\,\omega^2 + 5\,A_5\,\omega^4 - \ldots - n\,A_n\,\varphi_{n-4}\,(\zeta)\,\omega^{n-1} \right]$$

$$\frac{d\,B\,4}{d\,\omega} = \frac{1}{\varphi_4(\zeta)} \left[-\frac{3\,A_1}{\omega^4} + 2\,A_2\,\varphi_2\,(\zeta)\,\frac{1}{\omega^3} + A_3\,\varphi_3\,(\zeta)\,\frac{1}{\omega^2} - A_5\,\varphi_5(\zeta) - \ldots - (n-4)\,A_n\,\varphi_n\,(\zeta)\,\omega^{n-5} \right]$$

$$F(s) = s^3 + s^2 + 2s + 1 = (s + 1)(s^2 + s + 1)$$

$$s = -1, -\frac{1 \pm j\sqrt{3}}{2}$$

Example II

For a fourth order equation B_o vs B_1, $\zeta = 0$ and $\zeta = 0.2$.

$$F(s) = s^4 + 4.2\, s^3 + 6.6\, s^2 + B_1\, s + B_o$$

For $\zeta = 0$,

$$B_o = A_2\, \omega^2 - A_4\, \omega^4 \qquad (4\text{-}2)$$

$$B_1 = A_3\, \omega^2$$

The table indicates thatboth B_o and B_1 begin from zero; B_o ends at $-\infty$, whereas B_1 ends at $+\infty$; B_o has a maximum at $\omega = \sqrt{A_2/2A^4}$. Also equation (4-2) indicates that $B_o = 0$ at $\omega_o = \sqrt{A_2/A_4}$. Thus :

$$\omega_{max} \text{ for } B_o = \sqrt{6.6/2} = 1.8$$

$$\omega_o = \sqrt{6.6} = 2.6$$

$$B_o(1.8) = 10.9 \qquad B_1(1.8) = 13.9$$

$$B_o(2.6) = 0 \qquad B_1(2.6) = 27.7$$

The curves are shown in Fig. 4-2.

For $\zeta = 0.2$,

$$B_o = A_2\, \omega^2 - A_3\, \varphi_2(\zeta)\, \omega^3 - A_4\, \varphi_3(\zeta)\, \omega^4$$

$$B_1 = A_2\, \varphi_2(\zeta)\, \omega + A_3\, \varphi_3(\zeta)\, \omega^2 + A_4 \varphi_4(\zeta) \omega^3 \qquad (4\text{-}3)$$

The table's indications are: both B_o and B_1 are equal to zero at $\omega = 0$; both B_o and B_1 are equal to $-\infty$ at $\omega = \infty$, if $\zeta = 0.2$; each curve may have a maximum. The numerical values of the $\varphi(\zeta)$ functions are:

$$\varphi_2(0.2) = 0.4$$

$$\varphi_3(0.2) = 0.84$$

$$\varphi_4(0.2) = -0.736$$

The positive values of ω_{max} in the table are the only ones which can make B_o and B_1 maximum, and they are:

$$\omega_{max} \text{ for } B_o$$

$$= \frac{-3 \times 4.2 \times 0.4 + \sqrt{(3 \times 4.2 \times 0.4)^2 + 32 \times 6.6 \times o.84}}{8 \times 0.84} =$$

$$= 1.4$$

$$\omega_{max} \text{ for } B_1$$

$$= \frac{-4.2 \times 0.84 - \sqrt{(4.2 \times 0.84)^2 + 3 \times 6.6 \times 0.4 \times 0.736}}{-3 \times 0.74} =$$

$$= 3.5$$

The corresponding curves are shown in Fig. 4-3. Notice: all of the roots of F(s) have a damping ratio $\zeta \geq 0.6$, if $B_o = 2$, $B_1 = 5.4$. This point (5.4,2) is shown, in $B_o - B_1$, curves for $\zeta = 0$ and $\zeta = 0.2$, to be in the enclosure.

Example III

For a fifth order equation, B_o vs B_1, $\zeta = 0$

$$F(s) = s^5 + 7\,s^4 + 18\,s^3 + 23\,s^2 + B_1\,s + B_o$$

$$B_o = A_2\omega^2 - A_4\omega^4$$

$$B_1 = A_3\omega^2 - A_5\omega^4 \qquad (4\text{-}4)$$

According to the indications of the table, both B_o and B_1 are zero at $\omega = 0$; $B_o = B_1 = -\infty$ at $\omega = \infty$; each has a maximum at

$$\omega_{max} \text{ (for } B_o) = \sqrt{A_2/2A_4} = \sqrt{23/14} = 1.3,$$

$$\omega_{max} \text{ (for } B_1) = \sqrt{A_3/2A_5} = \sqrt{18/2} = 3.$$

Equation 4-4 shows that B_o is zero at $\omega_o = \sqrt{A_2/A_4} = 1.8$; B_1 is zero at $\omega_o = \sqrt{A^3/A_5} = 4.2$. Thus:

$B_o\,(1.3) = 18.9$	$B_1\,(1.3) = 26.8$
$B_o\,(3) = -360$	$B_1\,(3) = 81$
$B_o\,(1.8) = 0$	$B_1\,(1.8) = 48.3$
$B_o\,(4.2) = -1854$	$B_1\,(4.2 = 0$

The curves are shown in Fig. 4-4. Notice: All of the roots of F(s) have a damping ratio $\zeta \geq 0.5$ if

$$B_o = 6$$

$$B_1 = 17$$

This point (17,6) is shown in Fig. 4-4 to be in the stable domain.

Example IV

For a seventh order equation, B_o vs B_1, $\zeta = 0$

$$F(s) = s^7 + 9\,s^6 + 33\,s^5 + 66\,s^4 + 81\,s^3 + 63\,s^2 + B_1 s + B_o$$

$$B_o = A_2\omega^2 - A_4\omega^4 + A_6\omega^6$$

$$B_1 = A_3\omega^2 - A_5\omega^4 + A_7\omega^6 \qquad (4\text{-}5)$$

Table 4.1 shows that both B_o and B_1 are zero at $\omega = 0$; $B_o = B_1 = \infty$ at $\omega = \infty$; both can have one maximum and one minimum, but are not required to have such singular points other than at $\omega = 0$.

Table 4.2 shows:

$$\frac{d\,B_o}{d\omega} = 2A_2\omega - 4A_4\omega^3 + 6A_6\omega^5$$

$$\frac{d\,B_1}{d\omega} = 2A_3\omega - 4A_5\omega^3 + 6A_7\omega^5 \qquad (4\text{-}6)$$

For this case B_o' has three real roots;

$$\omega = 0,\ \omega^2 = \frac{132 \pm \sqrt{(132)^2 - 4 \times 27 \times 63}}{54} = 4.2 \text{ and } 0.55$$

$\omega = 0$, 0.74, and 2.05. B_1' also has three real roots; $\omega = 0$, 1.15, and 4.5.

The curves may have a shape as in Fig. 4-5. From these, the shape of $B_o - B_1$ is apparent from the sequence of frequency for the maximum, minimum and zero points as shown in Fig. 4-6. Note that either (a) or (b) sketches are possible, because the magnitudes of B_o and B_1 have not been calculated. If the critical values of ω are substituted into the B_o and B_1 equations, the coordinates of the points are determined and the sketch can be made accurate. Thus

ω	0.74	2.05	1.15	4.5
ω^2	0.55	4.20	1.33	20.7
ω^4	0.30	17.65	1.77	428
ω^6	0.16	74.1	2.35	8870

and the coordinates of the critical points are:

$$B_o(0.74) = 172 \qquad B_1(0.74) = 35$$

$$B_o(1.15) = -15 \qquad B_1(1.15) = 48$$

$$B_o(2.05) = -237 \qquad B_1(2.05) = -170$$

$$B_o(4.5) = 52905 \qquad B_1(4.5) = -3553$$

Thus, the B_o vs B_1 curve is shown to be as in Fig. 4-7. Notice: All of the roots of F(s) have a damping ratio $\zeta \geq 0.5$, if $B_1 = 29$, $B_o = 6$.

$$F(s) = s^7 + 9\,s^6 + 33\,s^5 + 66\,s^4 + 81\,s^3 + 63\,s^2 + 29\,s + 6$$

$$= (s + 1)^2(s +)^2\,(s + 3)(s^2 + s + 1).$$

This point (29.6) is shown to be inside the stable domain.

Example V

For the fourth order equation, B_2 vs B_3, $\zeta = 0$.

$$F(s) = s^4 + B_3\,s^2 + B_2\,s^2 + 5.4\,s + 2$$

$$B_2 = A_o\frac{1}{\omega^2} + A_4\,\omega^2$$

$$B_3 = A_1\frac{1}{\omega^2} \qquad (4\text{-}6)$$

The indications of the table are that both B_2 and B_3 are ∞ at $\omega = 0$; $B_2 = \infty$, $B_3 = 0$ at $\omega = \infty$; B_2 has a maximum at

$$\omega = (A_o/A_4)^{1/4} = (2)^{1/4} = 1.18$$

Thus:

$$B_2\,(1.18) = 2.82,$$

$$B_3\,(1.18) = 3.83$$

and the curves are shown in Figs. 4-8 and 4-9. Notice all of the roots of F(s) have a damping ratio $\zeta \geq 0.6$ is $B_2 = 6.6$, $B_3 = 4.2$

$$F(s) = s^4 + 4.2\,s^3 + 6.6\,s^2 + 5.48 + 2$$

$$= (s + 1)(s + 2)(s^2 + 1.2\,s + 1).$$

The point (4.2, 6.6) is shown in the graph to be lying in the stable domain.

Example VI

For a fifth order equation, B_1 vs B_2, $\zeta = 0$

$$F(s) = s^5 + 7s^4 + 18\, s^3 + B_2\, s^2 + B_1\, s + 6$$

$$B_1 = A_3\omega^2 - A_5\omega^4 \qquad (4\text{-}7)$$

$$B_2 = A_o \frac{1}{\omega^2} + A_4\omega^2$$

The table shows that $B_1 = 0$, $B_2 = +\infty$ at $\omega = 0$; $B_1 = -\infty$, $B_2 = +\infty$ at $\omega = \infty$; B_1 has a maximum, and B_2 has a minimum at:

$$\omega_{max} = \sqrt{A_3/2A_5} = \sqrt{18/2} = 3$$

$$\omega_{min} = (A_o/A_4)^{1/4} = (6/7)^{1/4} = 0.96$$

Equation 4-7 shows that B_1 is zero at

$$\omega_o = \sqrt{A_3/A_5} = 4.24$$

Thus, the critical points are at:

$$B_1(0.96) = 15.7 \qquad B_2\,(0.96) = 12.9$$

$$B_1(3) \quad = 63.6 \qquad B_2\,(3) \quad = 63.6$$

$$B_1(4.24) = 0 \qquad B_2\,(4.24) = 126.3$$

The curves are shown in Fig. 4-10

Example VII

For a fourth order equation, B_1 vs B_2, $\zeta = 0.6$

$$F(s) = s^4 + 4.6\, s^3\, B_2 s^2 + B_1 s + 2$$

$$B_1 = A_o\varphi_2(\zeta)\frac{1}{\omega} + A_3\omega^2 - A_4\,\varphi_2\,(\zeta)\,\omega^3 \qquad (4\text{-}8)$$

$$B_2 = A_o \frac{1}{\omega^2} + A_3\,\varphi_2(\zeta)\,\omega + A_4\,\varphi_3(\zeta)\,\omega^2$$

Table 4-2 gives the following equations, if the corresponding values of A_k and $\varphi_k\,(\zeta)$ are substituted;

$$\frac{d\,B_1}{d\omega} = \frac{1}{\omega^2}\,(2.4 - 9.2\,\omega^3 + 3.6\,\omega^4)$$

$$\frac{dB_2}{d\omega} = -\frac{1}{\omega^3}\,(4\text{-}5.52\,\omega^3 + o.88\,\omega^4) \qquad (4\text{-}9)$$

From the Table 4-1, it is seen that $B_1 = B_2 = +\infty$ at $\omega = 0$;

$B_1 = B_2 = -\infty$ at $\omega = \infty$. When $\zeta = 0.6$, the maximum or minimum frequency is not available through the routine methods. Even their existence is not readily checked.

Equation 4-9 can be used to improve the situation. It is seen from figure 4-11 that there may be two points where the difference of the curves is just 2.4, since by the differential calculus it is easily shown that the difference between the two curves attains its maximum value of 16.3 at $\omega = 1.91$. So there must be one frequency before 1.91, and another after 1.91, which realize these conditions. It is rather easy to find these values by trial and error, and they are found to be;

$$\omega = 0.7 \quad \text{and} \quad 2.5$$

By reasoning of the whole figure, the smaller one must be the minimum point, and the larger one the maximum.

Through the same analysis applied to the other equation of 4-9, it is seen that $\omega = 0.95$ and 6.2 are the required ones. Thus, this second relationship provides Fig. 4-12. B_1' and B_2' are then plotted as in Fig. 4-13 and the Mitrovic curve is sketched in Fig. 4-14

If the critical values of ω are substituted into equation (8), the graph becomes more accurate;

$$B_1(0.7) = 5.21 \qquad B_2(0.7) = 7.65$$

$$B_1(0.95) = 5.51 \qquad B_2(0.95) = 7.05$$

$$B_1(2.5) = 11.22 \qquad B_2(2.5) = 11.3$$

$$B_1(6.2) = -109 \qquad B_2(6.2) = 17.4$$

These are deduced by using the table;

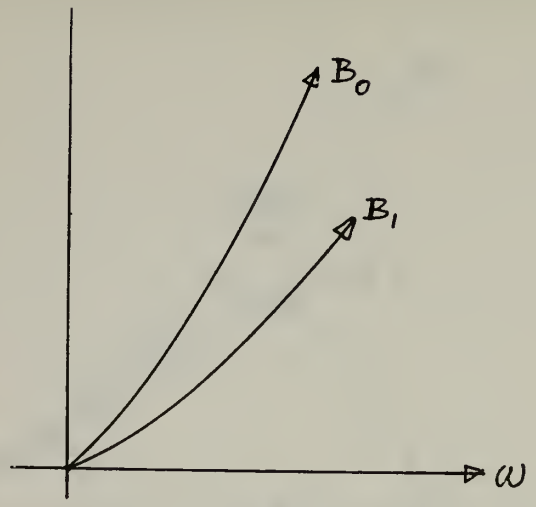
ω	0.7	0.95	2.5	6.2
ω^2	0.49	0.90	6.3	38.4
ω^3	0.34	0.86	15.6	238
$1/\omega$	1.4	1.0	0.4	0.161
$1/\omega^2$	2	1.1	0.157	0.026

A more accurate sketch of the B_1 vs B_2 curve is then available, as shown in Fig. 4-15. Notice: All of the roots of F(s) have a damping ratio $\zeta \geq 0.8$, if $B_1 = 6.2$, $B_2 = 7.8$.

$$F(s) = s^4 + 4.6\, s^3 + 7.8\, s^2 + 6.2\, s + 2$$

$$= (s + 1)(s + \;)(s^2 + 1.6\, s + 1).$$

This pair is shown to be in the enclosure, as should be.



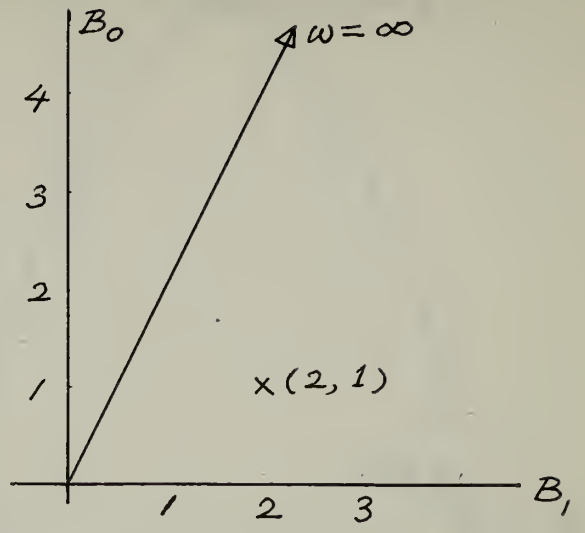


Figure 4-1

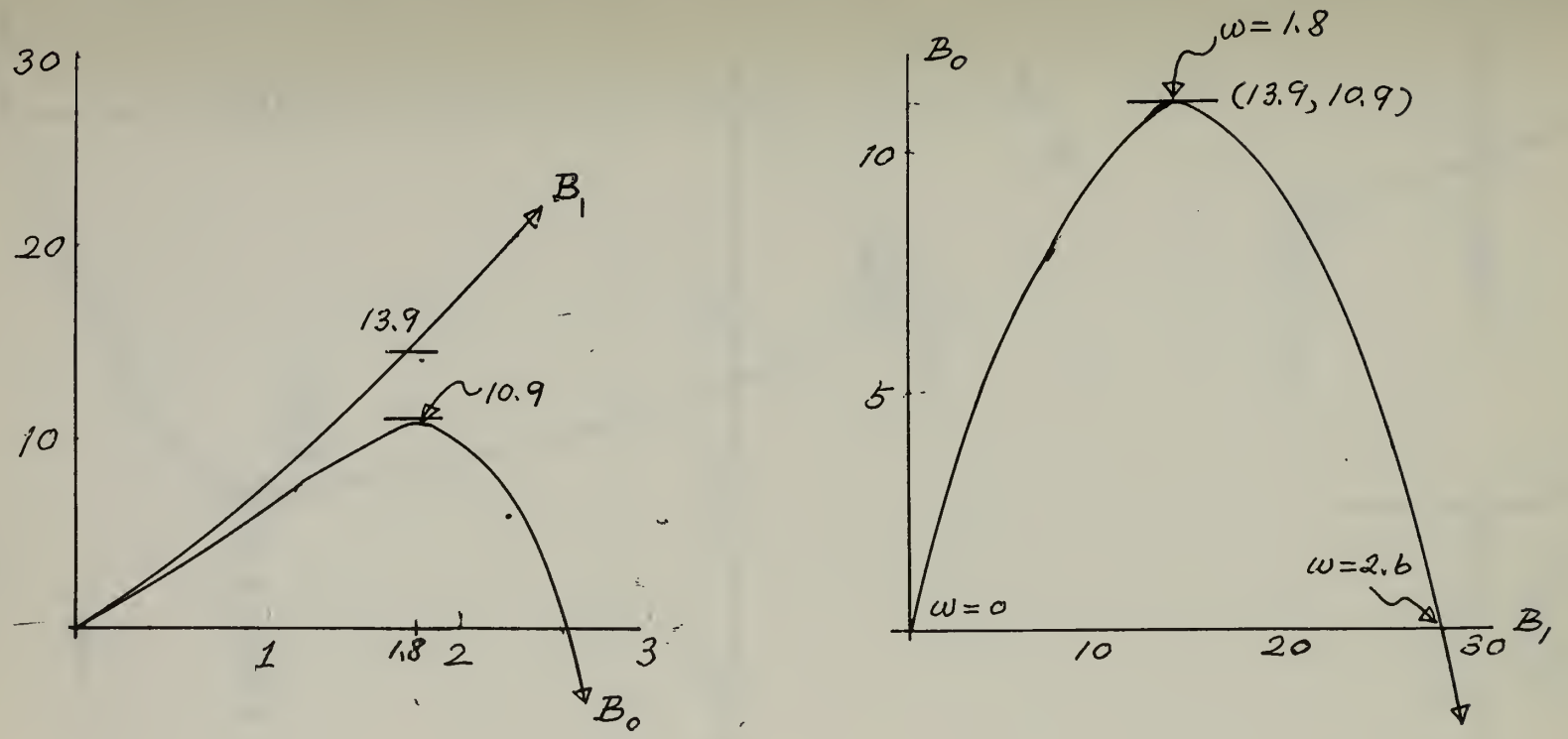

30
20
10
B_1
13.9
10.9
1
1.8
2
3
B_0
B_0
$\omega = 1.8$
(13.9, 10.9)
10
5
$\omega = 2.6$
$\omega = 0$
10
20
30
B_1

Figure 4-2

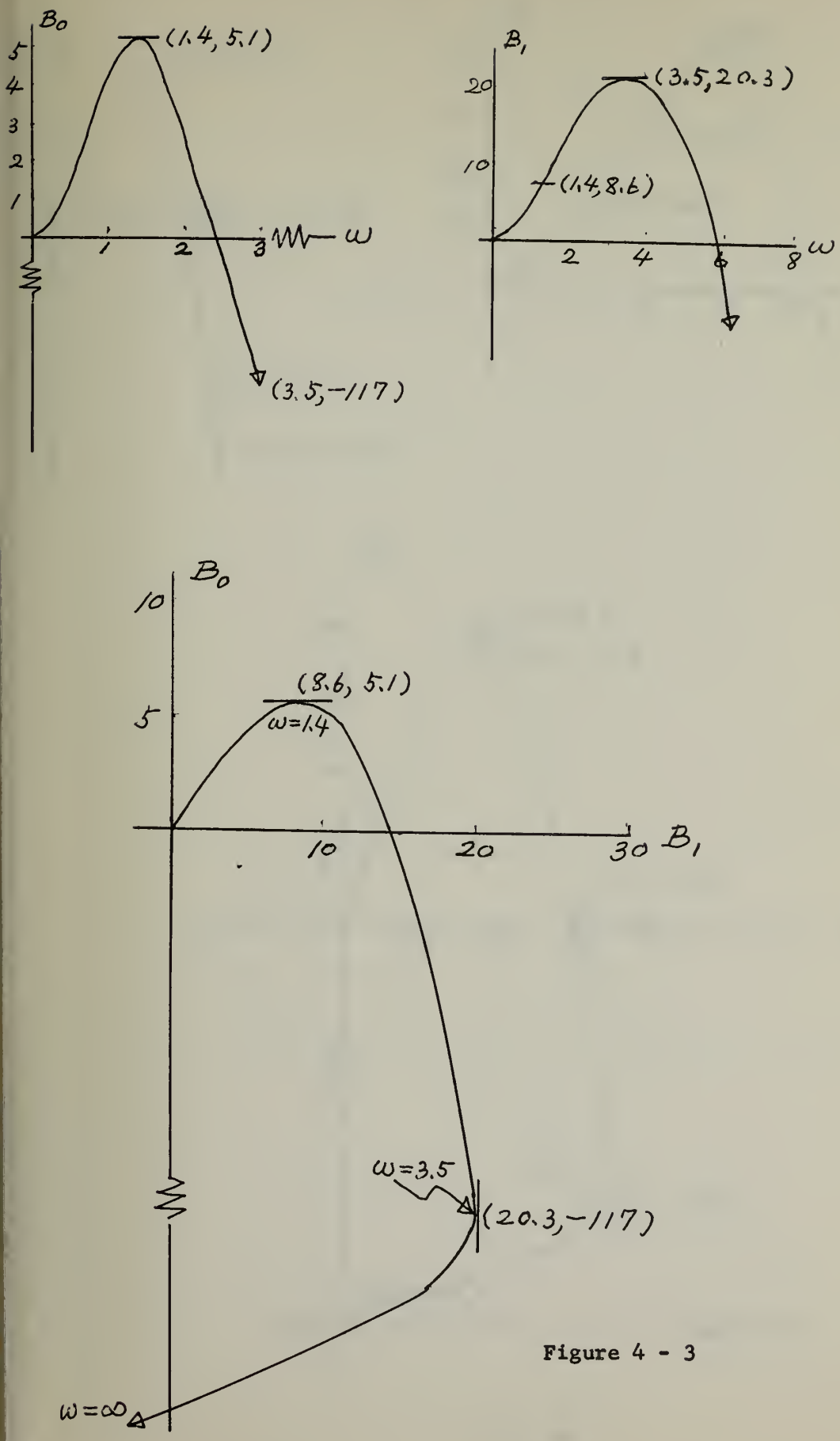

B_0
(1.4, 5.1)
5
4
3
2
1
1
2
3
ω
(3.5, −117)
B_1
20
10
(3.5, 20.3)
(1.4, 8.6)
2
4
6
8
ω
B_0
10
5
(8.6, 5.1)
ω=1.4
10
20
30
B_1
ω=3.5
(20.3, −117)
ω=∞

Figure 4 - 3

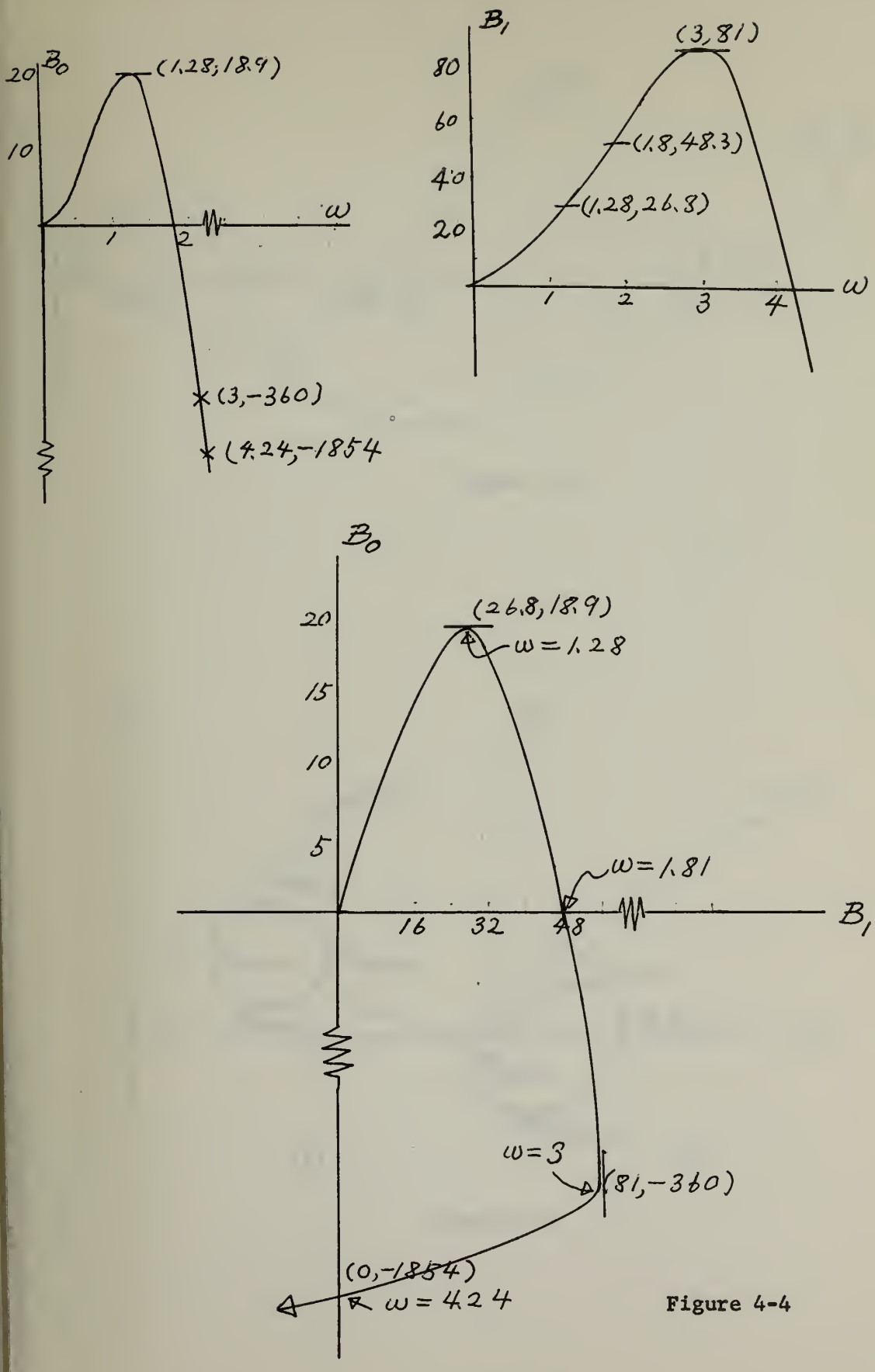

B_0
20
10
(1.28; 18.9)
ω
1
2
(3, −360)
(4.24, −1854
B_1
(3, 81)
80
60
40
20
(1.8, 48.3)
(1.28, 26.8)
1
2
3
4
ω
B_0
20
15
10
5
(26.8, 18.9)
ω = 1.28
ω = 1.81
16
32
48
B_1
ω = 3
(81, −360)
(0, −1854)
ω = 4.24

Figure 4-4

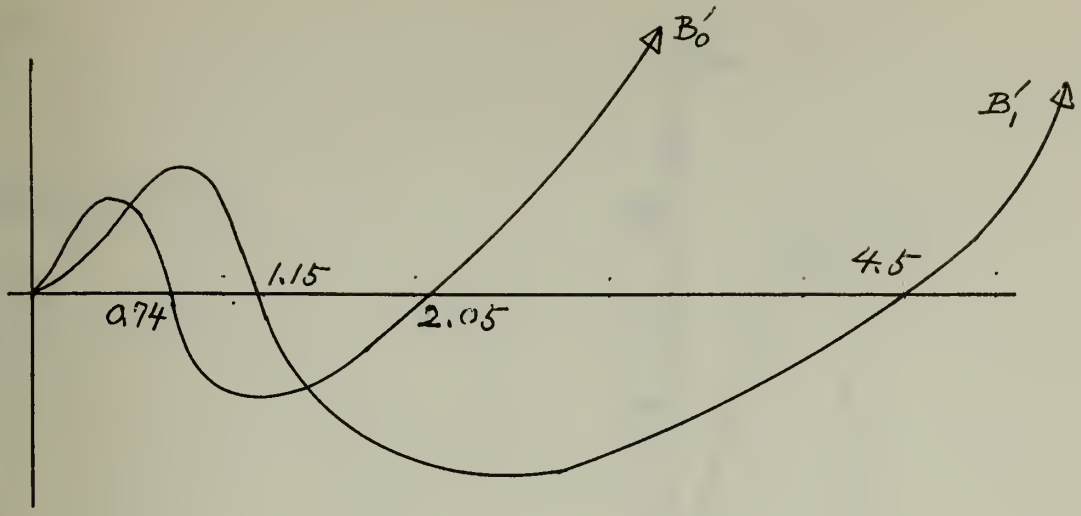


Figure 4-5

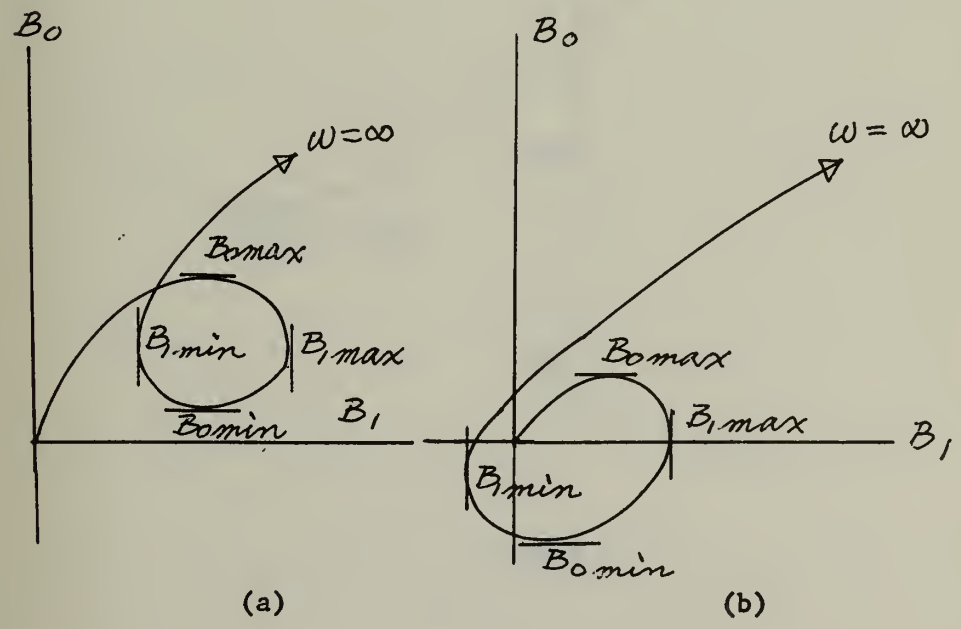


Figure 4-6

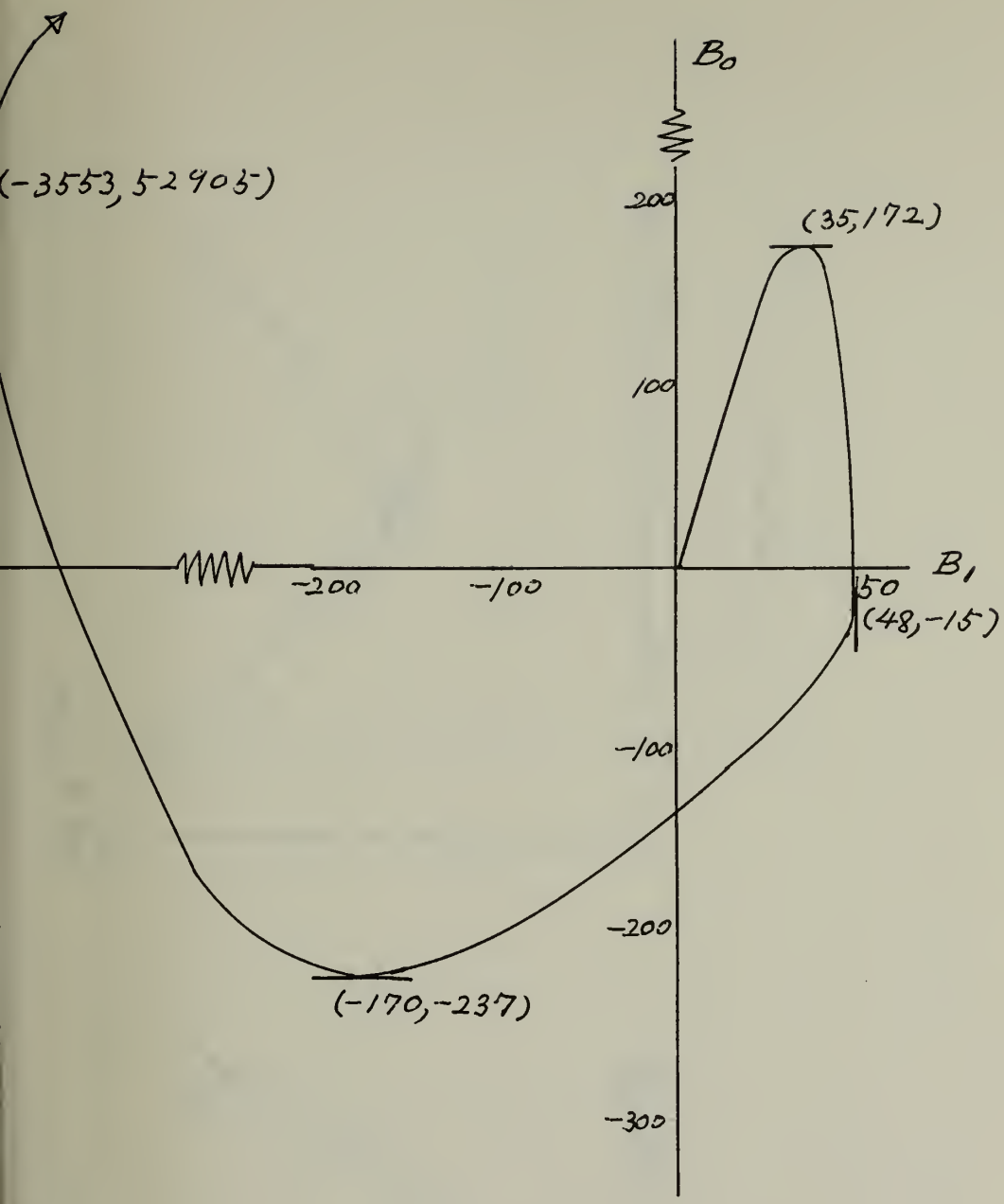

(-3553, 52903)
B_0
200
(35,172)
100
B_1
-200
-100
50
(48,-15)
-100
-200
(-170,-237)
-300

Figure 4-7

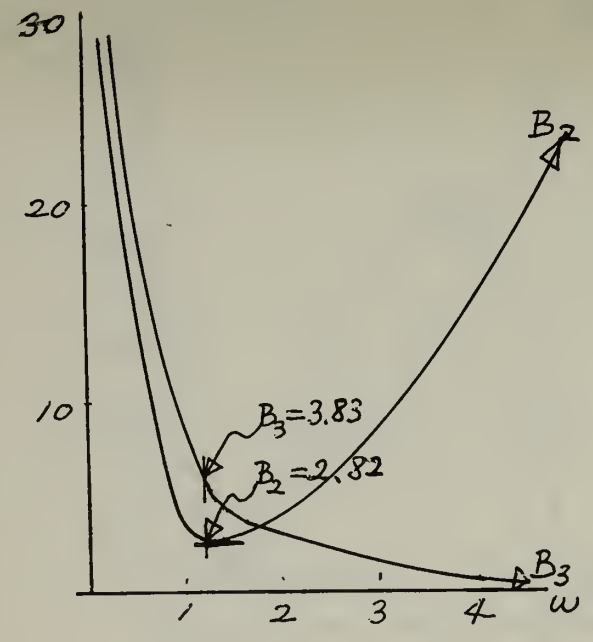


Figure 4-8

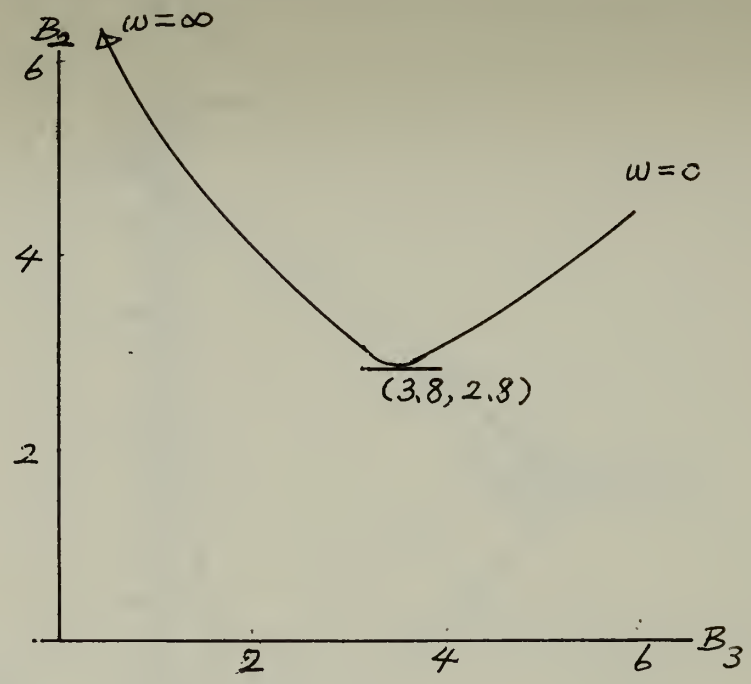


Figure 4-9

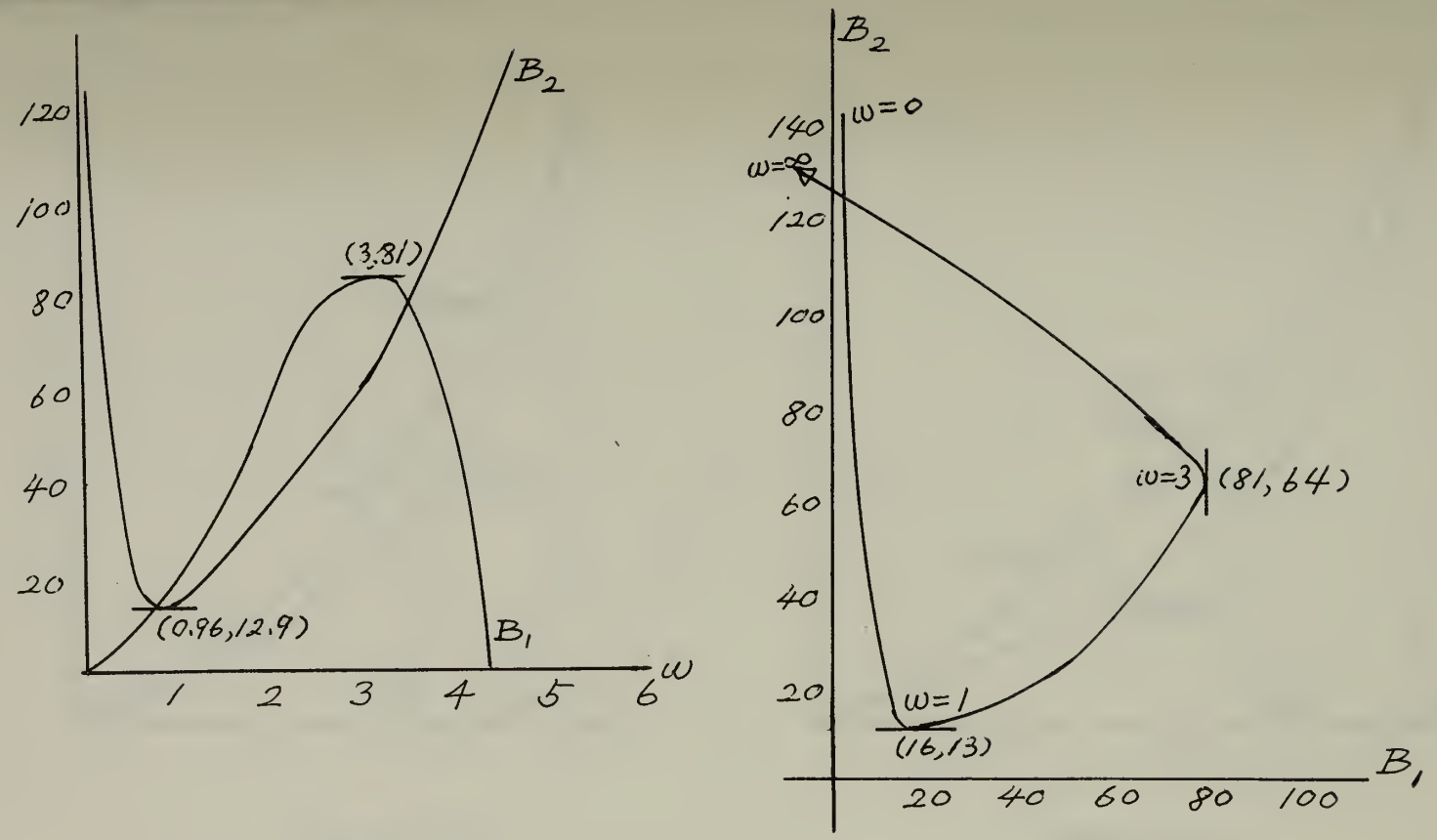

B_2
120
100
80
60
40
20
(3,81)
(0.96,12.9)
B_1
1
2
3
4
5
6
ω
B_2
140
$\omega=0$
$\omega=\infty$
120
100
80
60
40
20
$\omega=3$
(81, 64)
$\omega=1$
(16,13)
B_1
20
40
60
80
100

Figure 4-10

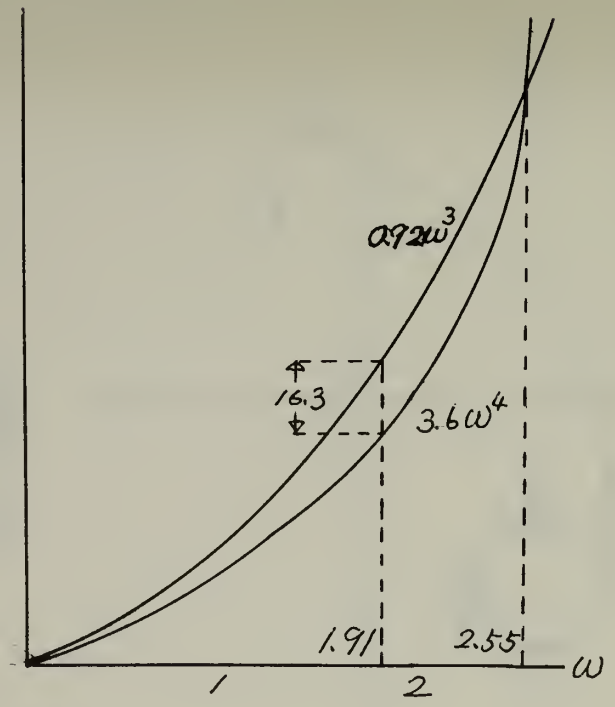


Figure 4-11

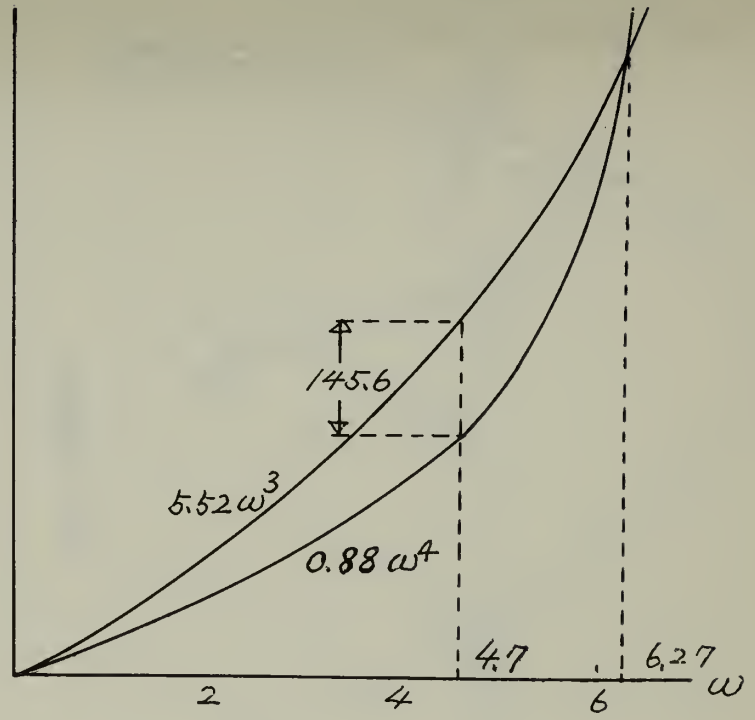


Figure 4-12

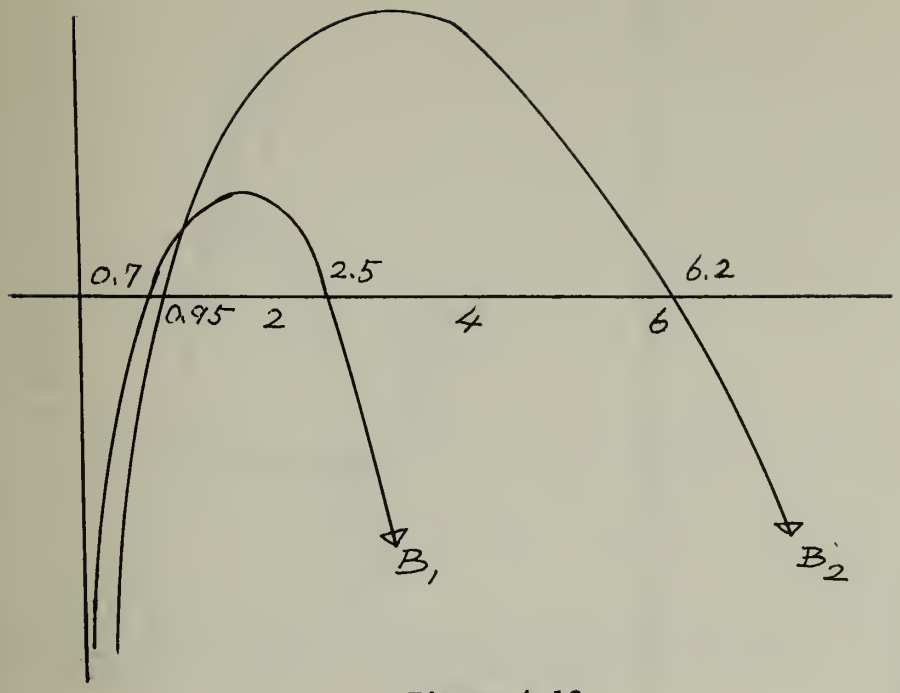


Figure 4-13

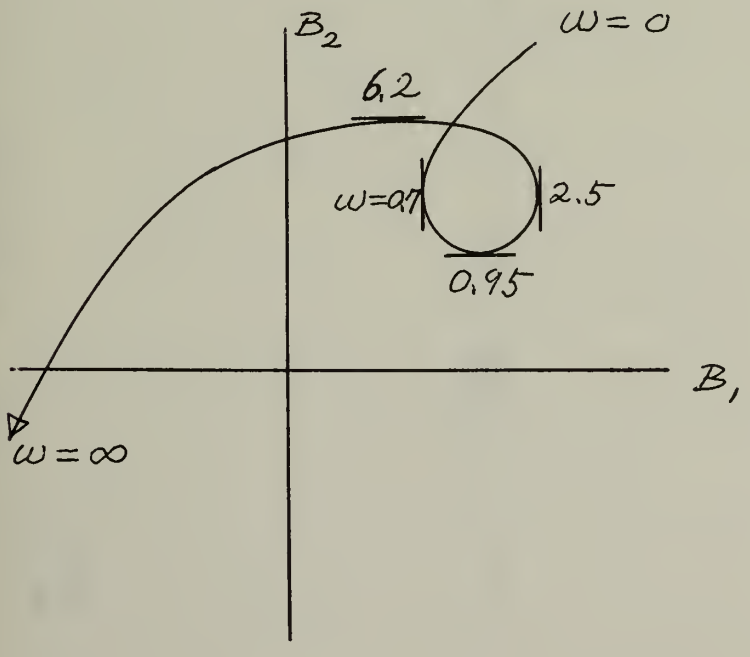


Figure 4-14

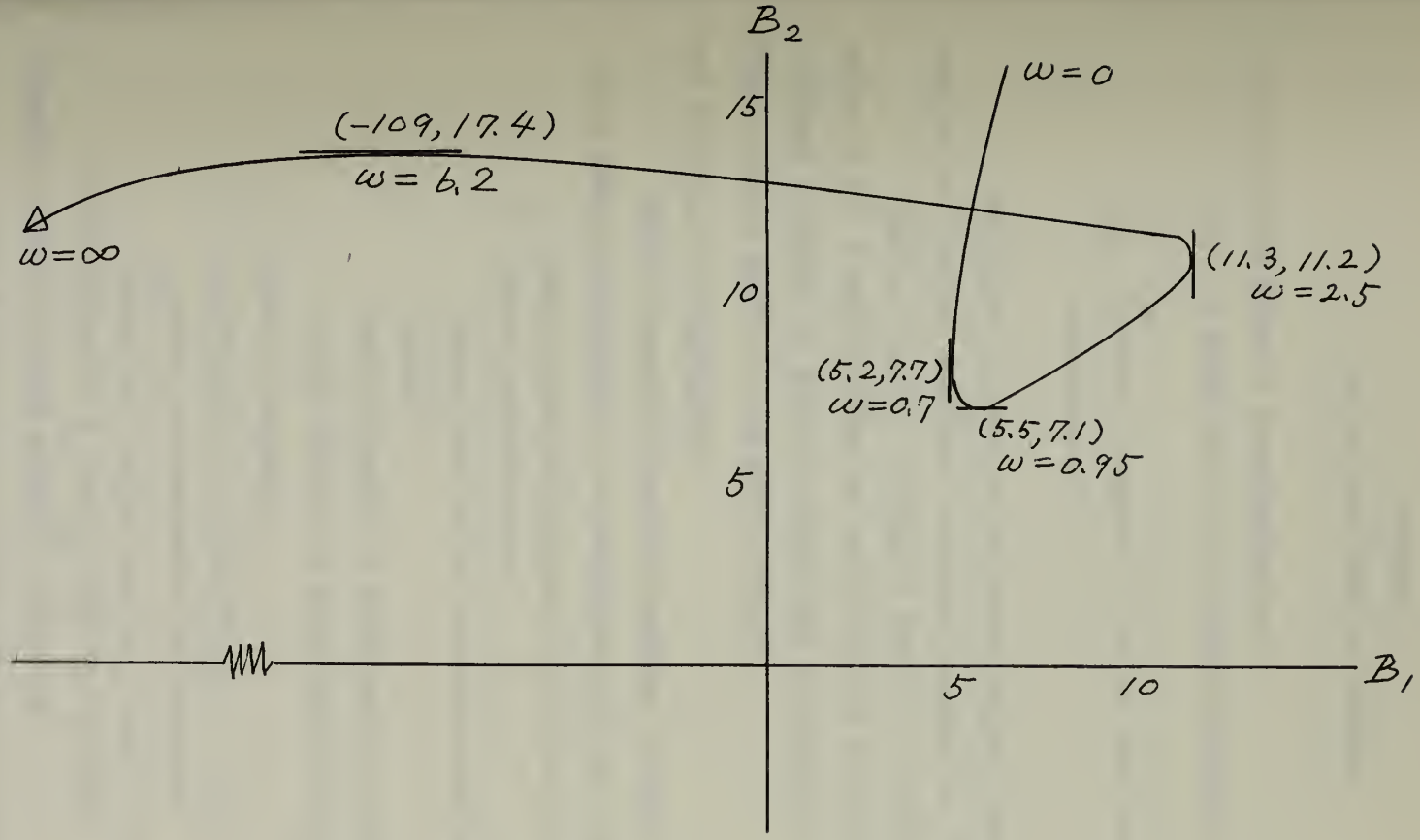

B_2
15
10
5
B_1
5
10
$\omega = 0$
(-109, 17.4)
$\omega = 6.2$
$\omega = \infty$
(11.3, 11.2)
$\omega = 2.5$
(5.2, 7.7)
$\omega = 0.7$
(5.5, 7.1)
$\omega = 0.95$

Figure 4-15

CHAPTER V

THE DIGITAL COMPUTER PROGRAM

The aims of the present programming are:

(i) To print out the calculated values of any pair of B_e and B_m:

(ii) To make a graphical plot of this pair on section papers for each value of ζ.

The first is done by the usual digital machine, and the second is done by the digital analogue converter attached to the system.

Some safe-guard statements had to be included in the programming owing to the generalization of applicability to any pair, in addition to the original purposes of getting graphs and calculated results for varying ζ. These will be shown in the next section.

The practice of programming:

What is needed to run the machine is just to punch the cards according to the indication shown in the beginning part of the programming:

1) User's name and program identification:

2) Output specification,

Blank card is for the case both print and graph are required.

Punch "graph" if graph only is required.

Punch "print" if print only is required.

3) Order of equation.

4) Coefficients of equation, A_o ..., A_n, including the A_e and A_m which are assumed to be variable.

5) Values of e and m, such as 1 and 2 for B_1 - B_2 case.

The values of B_e and B_m vary in two ways: one, through the changing value of ζ, and another, through ω.

Usually there are no difficulties in changing the values of ζ from 0 to 1 by an arbitrary chosen step if the difference of e and m is not even. If it is even, B values become infinite and the machine stops, as ζ takes the value zero. The same situation occurs if the difference is a multiple of three when ζ is equal to 0.5. These difficulties are auto-

matically avoided by making the machine choose the adequate starting values and step width of ζ in the programming.

The problem of how to find out the proper upper limit of ω comes next. It seems very logical to set the limit around the value of ω which makes the last term of each B dominate, or become larger than the sum of other terms, since this actually assures that the curve $B_e - B_m$ is plotted into the last quadrant where it ends at ω = infinity. This is also necessary from the practical point: It sometimes occurs in certain pairs of B_e and B_m with certain values of ζ that the curve re-appears in the first quadrant at the larger frequencies and divides the enclosure or stable domain, and changes a part of it to lie out of enclosure or unstable. Rather a lengthy part is spent in the programming to secure the effective results. But in many cases, the plotted graph of $B_e - B_m$ becomes inaccurate by extending the value of ω this much, though the graph scaling is automatically set for each value of ζ in order to get a full resolution, since the values of them at the start and at the end differ tremendously. On this occasion, it is rather recommended to try to set the upper limit of ω to just half of the value indicated by the machine. This decreases the values of the highest B_e and B_m to nearly one-sixteenth in the case of fourth order equations. It is a good practice to decrease the values of ω after getting the printed results to get graphs with better resolution whenever it is necessary. It also improves the starting part of the printed results, and thus sometimes shows the maximum or minimum at the very lower frequencies hidden in the original setting. Anyhow, it is necessary to know the tendency of the curve at $\omega = 0$ and at $\omega = \infty$ for correct understanding of the results, and it is very easy to get these knowledges by a glance at the $B_e - B_m$ equations.

To improve the printed results is easily done by omitting certain statements which are inserted to make the machine print out only every tenth out of the 900 calculated results in the memory box required for the graph.

It may be said that many practical applications come out of this programming, and if some applications are beyond the present programming, it may be easily attained by just manipulating a few statements: For an example, if the behavior of the curve of $B_2 - B_o$ at $\zeta = 0$ is needed, it is

easily attained by making the starting value of ζ negative and the ending value positive just around the origin and thus getting the full account of the curve tendency there. Actually just two statements, namely, statement 74 and 49 are to be manipulated in this case.

CHAPTER VI

A PROCEDURE FOR COMPENSATION DESIGN

6.1 Cascade Compensation

Throughout the procedure of compensation design given here, K_v, K_a, or K_p is assumed to be the same as the original system.

It is assumed that a system with a transfer function

$$G_u(s) = \frac{K_o \prod_{i=1}^{m}(s + z_i)}{s^N \prod_{i=1}^{n}(s + p_i)} \qquad (6\text{-}1)$$

is to be compensated by a filter with

$$G_c(s) = \frac{K_c \quad s + z}{s + p} \qquad (6\text{-}2)$$

to have $\zeta \geq \zeta_o$. Then the characteristic equation is

$$s^{N+n+1} + A_{N+n}(P)\, s^{N+n} + \ldots + A_{m+1}(P, K_c) s^{m+1} + \ldots$$

$$+ A_1(P, K_c)\, s + A_o(P, K_c) = 0 \qquad (6\text{-}3)$$

Actually the last term is

$$A_o(P, K_c) = K_o \cdot P$$

as K_v's are constant.

It is very remarkable that any of $A_\alpha(P)$ and $A_\beta(P, K_c)$ is a linear function of P and P and K_c. (This can be seen easily.) This implies that the gradient of B_e or B_m has some value independent of the value of P, determined only by values of ζ and ω_n, and B_e and B_m are both linearly expressed in A_α and A_β. This is seen as follows:

$$B_e(P, K_c) = \sum_{i,j,k} \varphi_i(\zeta)\, \omega_n^j\, A_k(P, K_c)$$

$$B_m(P, K_c) = \sum_{a,b,c} \varphi_\alpha(\zeta)\, \omega_n^b\, A_c(P, K_c) \qquad (6\text{-}4)$$

The effect of the change of P and K_c values, fixing ζ and ω_n is shown as:

$$\Delta B_e = \sum_{i,j,k} \varphi_i(\zeta)\, \omega_n^j \left[\frac{\partial A_k}{\partial p} \Delta P + \frac{\partial A_k}{\partial K_c} \Delta K_c \right]$$

$$\Delta B_m = \sum_{a,b,c} \varphi_a(\zeta)\, \omega_n^b \left[\frac{\partial A_c}{\partial p} \Delta P + \frac{\partial A_c}{\partial K_c} \Delta K_c \right]. \qquad (6\text{-}5)$$

So A_k and A_c are all linear in P and K_c, it is seen that both ΔB_e, and ΔB_m depend only on the differences of P and K_c.

Also it is seen that

$$\Delta A_e = \frac{\partial A_e}{\partial p} \Delta P + \frac{\partial A_e}{\partial K_c} \Delta K_c$$
$$\Delta A_m = \frac{\partial A_m}{\partial p} \Delta P + \frac{\partial A_m}{\partial K_c} \Delta K_c \qquad (6\text{-}6)$$

If an arbitrarily chosen P and K_c make the M (A_e, A_m) point lie on or inside the Mitrovic curve B_e - B_m with specified $\zeta = \zeta_o$, a desired damping has been obtained and the design procedure ends. But this seldom occurs in involved problems. In such cases P and K_c must be varied so as to make a new M' (A_e', A_m') point lie on or inside the new B_e' - B_m' curve. This is attained by making, at a certain value of ω_n, say at $\omega_n = \omega_o$, M' locus cut B_e' - B_m' curve. This is mathematically shown as

$$A_e' = B_e'$$
$$A_m' = B_m' \qquad (6\text{-}7)$$

where

$$A_e' = A_e + \frac{\partial A_e}{\partial p} \Delta P + \frac{\partial A_e}{\partial K_c} \Delta K_c$$
$$A_m' = A_m + \frac{\partial A_m}{\partial p} \Delta P + \frac{\partial A_m}{\partial K_c} \Delta K_c$$
$$B_e' = B_e + \frac{\partial B_e}{\partial p} \Delta P + \frac{\partial B_e}{\partial K_c} \Delta K_c \qquad (6\text{-}8)$$
$$B_m' = B_m + \frac{\partial B_m}{\partial p} \Delta P + \frac{\partial B_m}{\partial K_c} \Delta K_c$$

All A_e, A_m, B_e and B_m are previously calculated numbers. Trial values of P and K_c determine them. Substituting (8) into (7),

$$A_e - B_e = \Delta P \left(\frac{\partial B_e}{\partial p} - \frac{\partial A_e}{\partial p} \right) + \Delta K_c \left(\frac{\partial B_e}{\partial K_c} - \frac{\partial A_e}{\partial K_c} \right)$$
$$A_m - B_m = \Delta P \left(\frac{\partial B_m}{\partial p} - \frac{\partial A_m}{\partial p} \right) + \Delta K_c \left(\frac{\partial B_m}{\partial K_c} - \frac{\partial A_e}{\partial K_c} \right) \qquad (6\text{-}9)$$

From this ΔP and ΔK_c are given by

$$\Delta P = \frac{D_p}{D}$$

$$\Delta K_c = \frac{D_{Kc}}{D} \quad , \tag{6-10}$$

where

$$D = \left(\frac{\partial B_e}{\partial p} - \frac{\partial A_e}{\partial p}\right)\left(\frac{\partial B_m}{\partial K_c} - \frac{\partial A_e}{\partial K_c}\right) - \left(\frac{\partial B_m}{\partial p} - \frac{\partial A_m}{\partial p}\right)\left(\frac{\partial B_e}{\partial K_c} - \frac{\partial A_e}{\partial K_c}\right)$$

$$D_p = C_e\left(\frac{\partial B_m}{\partial K_c} - \frac{\partial A_m}{\partial K_c}\right) - C_m\left(\frac{\partial B_e}{\partial K_c} - \frac{\partial A_e}{\partial K_c}\right)$$

$$D_{Kc} = C_m\left(\frac{\partial B_e}{\partial p} - \frac{\partial A_e}{\partial p}\right) - C_e\left(\frac{\partial B_m}{\partial p} - \frac{\partial A_m}{\partial p}\right)$$

$$C_e = A_e - B_e$$

$$C_m = A_m - B_m \quad .$$

It must be remembered that all of the terms in any bracket are numbers as ζ and ω_n are specified.

Finally, if the newly found P' and K_c';

$$P' = P + \Delta P$$

$$K_c' = K_c + \Delta K_c$$

are physically acceptable, namely both P' and K_c' are positive and the magnitude of K_c is reasonable, then the design is completed and z is found to be $z = P'/K_c'$. If the solution is not acceptable this simply implies that a multiple section filter must be used.

If this happens, one tries to find values of P and K_c as favorable as possible for the first stage. The next stage is just a repetition of the procedure given here. This process can be carried on until the specifications are met.

It also must be noticed that although the above mentioned method is very general, it is sometimes found that a simpler treatment is applicable and preferable.

If the order of the whole compensated system is high so as to have more than one pair of complex conjugate roots whose real parts are in the same order of magnitude, the above mentioned procedure must be followed rather strictly.

But if it is known that there is just one pair of complex conjugate roots, or of dominant complex conjugate roots, then more simplified procedure is applicable.

In this occasion it is justified to locate M (A_e, A_m) point on any part of the B_e - B_m curve in the first quadrant. It is not necessary to place them on or inside the enclosure part of the curve. Then the problem degenerates to just finding out the values of P and K_c which makes

$$A_e = B_e$$
$$A_n = B_m \qquad (6\text{-}11)$$

A_k (P, K_c) takes the following form:

$$A_k\ (P,\ K_c) = \alpha_k\ P + \beta_k\ K_c + \gamma_k, \qquad (6\text{-}12)$$

where α_k, β_k, and γ_k are all given real numbers.

If (12 and (4) are substituted into (11) it gives:

$$P\ (\alpha_e - \Sigma \varphi_i(\zeta)\ \omega_n^j\ \alpha_k) + K_c\ (\beta_e - \Sigma \varphi_i(\zeta)\ \omega_n^j\ \beta_k)$$
$$= \Sigma \varphi_i\ (\zeta)\ \omega_n^j\ \gamma - \gamma_e$$
$$P\ (\alpha_m - \Sigma \varphi_a(\zeta)\ \omega_n^b\ \alpha_c) + K_c\ (\beta_m - \Sigma \varphi_a(\zeta)\ \omega_n^b\ \beta_c)$$
$$= \Sigma \varphi_a(\zeta)\ \omega_n^b\ \gamma_c - \gamma_m \qquad (6\text{-}13)$$

If ζ and ω_n are specified by some design requirements, all coefficients are fixed in (6-13) and then equations become simultaneous equations of P and K_c:

$$L_1\ P + L_2\ K_c = N_e$$
$$M_1\ P + M_2\ K_c = N_m \qquad (6\text{-}14)$$

whose solution is

$$P = \frac{N_e\ M_2 - N_m\ L_2}{L_1\ M_2 - L_2\ M_1}$$
$$K_c = \frac{N_m\ L_1 - N_e\ M_1}{L_1\ M_2 - L_2\ M_1} \qquad (6\text{-}15)$$

Here

$$L_1 = (\alpha_e - \Sigma \varphi_i(\zeta)\ \omega_n^j\ \alpha_k)$$

$$L_2 = (\beta_e - \Sigma \varphi_i(\zeta)\ \omega_n^j\ \beta_k)$$

$$M_1 = (\alpha_m - \Sigma \varphi_a(\zeta)\ \omega_n^b\ \alpha_c) \qquad (6\text{-}16)$$

$$M_2 = (\beta_m - \Sigma \varphi_a(\zeta)\ \omega_n^b\ \beta_c)$$

$$N_e = (\Sigma \varphi_i\ (\zeta)\ \omega_n^j\ \gamma_k - \gamma_e)$$

$$N_m = (\Sigma \varphi_a(\zeta)\ \omega_n^b\ \gamma_c - \gamma_m)$$

If the values found for P and K_c are physically realizable, then the design is completed. If it is not the case, then ζ or ω_n may be varied to find the acceptable values, in case such change of values of ζ or and ω_n are permitted. If it is not permitted, then multisection compensation is needed.

Example 6-1

$$G_u(s) = \frac{420}{s(s + 1)(s + 15)}$$

is to be compensated to have $\zeta \geq 0.6$ by a cascade filter.

Solution:

$$G_c(s) = \frac{K_c(s + z)}{(s + P)}$$

Characteristic equation is:

$$s^4 + (16 + P)\ s^3 + (15 + 16\ P)\ s^2 + (420\ K_c + 15\ P)\ s + 420\ P = 0$$

B_1 - B_2 equations for $\zeta = 0.6$ are:

$$B_1 = \frac{1.2}{\omega_n}\ (420\ P) + (16 + P)\ \omega_n^2 - 1.2\ \omega_n^3$$

$$B_2 = \frac{420\ P}{\omega_n^2} + 1.2\ (16 + P)\ \omega_n - 0.44\ \omega_n^2$$

$$A_1 = 15\ P + 420\ K_c$$

$$A_2 = 15 + 16\,P$$

$\omega_n = 0.7$ is tried:

$$A_1 = B_1 \rightarrow \qquad 420K_c - 705.7\,P = 7.58$$

$$A_2 = B_2 \qquad\qquad + 825\,P = 2.8$$

$$P = 0.0034$$

$$K_c = 0.024$$

$$z = \frac{P}{K_c} = 0.141$$

Thus the required $G_c(s)$ has the form

$$G_c(s) = \frac{0.024(s + 0.141)}{(s + 0.0034)}$$

The root locus is seen in Fig. 6-2. The points encircled by triangles are the roots of the system. This shows the applicability of the procedure.

6.2 Cascade Compensation (Continued)

For many problems it is more convenient to use less sophisticated design techniques, such as a combination of sketches with basic graphical interpretations, or curves calculated with the digital computer. Illustrations of these techniques are given in the following section.

Example 6.2

$$G_u(s) = \frac{106}{s\,(s + 5)}$$

Design a lead compensation which makes the system have $\zeta \geq 0.6$.

Solution:

$$G(s) = \frac{106.K_c(s + z)}{s(s + 5)(s + P)}$$

The characteristic equation is

$$s^3 + (5 + P)s^2 + (5P + 106\,K_c)s + 106\,K_c z = 0.$$

Mitrovic equations for $B_o - B_1$ scheme are, for this system:

$$B_o = (5 + P)\,\omega^2 - 1.2\,\omega^3$$

$$B_1 = 1.2(5 + P)\,\omega - 0.44\,\omega^2 .$$

Here $\varphi_k(\zeta)$ are expressed numerically for the value $\zeta = 0.6$. Previous tables tell that the figure of $B_o - B_1$ is like Fig. 6-2 to have an enclosure. It is actually seen that

$$\omega_{m1} < \omega_{m2}$$

holds for any P by the table. Also they show that

$$B_{o\ max} = \frac{1}{10}(5 + P)^3.$$

If $A_o = 106\ K_c z = 106\ P$ is in or on this enclosure, the following relation must exist:

$$106\ P \leq \frac{1}{10}(5 + P)^3.$$

$P \geq 25$, and $0 \leq P < 0.1$ satisfy this inequality. As a load section is required, $P = 28$ is adopted. Then $A_o = 2968$. B_o is equal to this value if $\omega \approx 23$. Substituting this value and $P = 28$ in the B_1 equation, it is seen

$$B_1 = 670$$

A_1 must have this value if $M(A_o, A_1)$ is on the $\zeta = 0.6$ curve.

$$A_1 = 5\ P + 106\ K_c = 670$$

which gives

$$K_c = 5,$$

and

$$z = P/K_c = 5.6.$$

Thus $G_c(s)$ turns out to be

$$G_c(s) = \frac{5\ (s + 5.6)}{(s + 28)}$$

If a lag network is desired, $P = 0.1$ is a good choice.

$$A_o = 106\ P = 10.6$$

B_o has this value when $\omega \approx 2$. B_1 has the value 10.48 correspondingly.

$$A_1 = 5\ P + 106\ K_c = 10.48$$

$$K_c \approx 0.1$$

$$z = P/K = 1$$

Thus the lag compensation has

$$G_c(s) = \frac{0.1\ (s + 1)}{(s + 0.1)}.$$

Example 6.3

$$G_u(s) = \frac{8\ (s + 6)}{s(s + 1)(s + 3)}$$

Stabilize this system with a single section compensation circuit.

Solution:

$$G_c(s) = \frac{K_c(s + z)}{(s + p)}$$

The characteristic equation for the whole system is

$$s^4 + (4 + P)\, s^3 + (3 + 4P + 8K_c)\, s^2 + \left\{3P + (6 + z)\, 8K_c\right\}$$

$$s + 48\, K_c z = 0.$$

The $B_1 - B_2$ plane for $\zeta = 0$ is chosen and the Mitrovic equations are

$$B_1 = (4 + P)\, \omega^2$$

$$B_2 = 48\, P/\omega^2 + \omega^2$$

$K_c z = P$ is the relation to maintain the error coefficient as before. From the tables given previously, it is seen the graph of $B_1 - B_2$ has the shape shown in Fig. 6-3. Also the tables give

$$B_{1\,min} = 4\,(4 + P)\sqrt{3\,P}$$

$$B_{2\,min} = 8\sqrt{3P}$$

Polar analysis, for which explanation is already given, tells that the M (A_1, A_2) point with the following relation

$$A_1 = B_{1m}$$

$$A_2 \geq B_{2m}$$

lies in the stable domain.

These equations are transformed,

$$11\, P + 48\, K_c = 4\,(4 + P)\sqrt{3P}$$

$$3 + 4P + 8\, K_c \geq 8\sqrt{3P}$$

Multiplying the inequality by 6, subtract the former equation from it, and the following inequality is derived:

$$(18 + 3P) > \sqrt{48P}\,(8 - P)$$

$P \geq 4$ and $P \leq 0.1$ satisfy this relation. Here it must be noticed that too large and too small values of P tend to make the corresponding value of K_c impractical. In this case, $P = 8$ and $P = 0.1$ are found to be good trial values.

(i) $P = 8$

$$B_{1m} = 12 \times \sqrt{384} = 235.2$$

$$A_1 = 88 + 48\, K_c = 235.2$$

$$K_c = \frac{147.2}{48} = 3.06$$

$$A_2 = 35 + 8K_c = 59.5$$

$$B_{2m} = 2\sqrt{48P} = 39.2 < A_2$$

Thus M (A_1, A_2) surely lies in the stable domain. $G_c(s)$ is found to be

$$G_c(s) = \frac{3.06\ (s + 2.56)}{(s + 8)} \quad ,$$

which is a lead network.

(ii) $P = 0.1$

By similar calculations a lag network with the following transfer function is obtained:

$$G_c(s) = \frac{0.17\ (s + 0.61)}{(s + 0.1)}$$

Example 6.4

$$G_u(s) = \frac{8(s + 6)}{s(s + 1)(s + 3)}$$

is to be compensated to have $\zeta \geq 0.4$ by a single lead network.

Solution:

$$B_1 = 0.8\, \frac{A_o}{\omega} + A_3\, \omega^2 - 0.8\, \omega^3$$

$$B_2 = \frac{A_o}{\omega^2} + 0.8\, A_3 \omega + 0.36\, \omega^2$$

First, assume $P = 15$, as was done in the previous problem. Comparing coefficients

$$B_1 = \frac{576}{\omega} + 19\, \omega^2 - 0.8\, \omega^3$$

$$B_2 = \frac{720}{\omega^2} + 15.2\, \omega + 0.36\, \omega^2.$$

These equations tell that the B_1 - B_2 curve starts at infinity in the first quadrant and ends at infinity in the second quadrant, as is seen in Fig. 6-4. From the polar plot of Fig. 6-5, it is easily seen that the enclosure exists only when the B_1 - B_2 curve takes the shape shown in Fig. 6-4. This means B_1 is to have one minimum and one maximum, while

while B_2 has just one minimum. If tables in THALER & ROWN: Feedback Control Systems are used, this calculation is easily done and the shape of the curve is shown as follows in Fig. 6-6. The digital computer does the same thing if the attached programming is used properly.

Equation (11) shows:

$$A_1 = 11\ P + P + 48\ K_c$$

$$A_2 = 3 + 4P + 8\ K_c \quad .$$

This A_1 is equated to $B_1 = 600$ to give the value of K_c,

$$K_c \approx 9.1$$

This gives the value of A_2

$$A_2 = 63 + 72.8 = 135$$

which surely lies in the $\zeta = 0.4$ enclosure. Thus the required circuit is

$$G_c = \frac{9.1\ (s + 1.65}{(s + 15)}$$

Example 6-5.

$$G_u(s) = \frac{420}{s(s + 1)(s + 15)}$$

is to be stabilized by a single section compensation circuit.

Solution:

$$G(s) = \frac{420\ K_c(s + z)}{s(s + 1)(s + 15)(s + P)}$$

is assumed to be the transfer function of the stabilized system. Then the characteristic equation for the system is

$$s^4 + (16 + P)s^3 + (15 + 16\ P)s^2 + (15\ P + 420\ K_c)s$$

$$+ 420\ K_c z = 0$$

As A_o is determined by P only, the B_1 - B_2 scheme is preferable.

The tables give the following equations for $\zeta = 0$:

$$B_1 = A_3\ \omega^2$$

$$B_2 = A_o/\omega^2 + \omega^2$$

and the shape of the curve is shown in Fig. 6-7. Also the tables tell that

$$B_2(\omega_{min}) = 2\sqrt{A_o}$$

$$B_1(\omega_{min}) = A_3\sqrt{A_o}\ .$$

The polar analysis indicates that the M point with

$$A_1 = B_1(\omega_{min}) = B_{1\ min}.$$

$$A_2 \geq B_{2\ min}$$

certainly lies in the stable domain.

These equations can be transformed into

$$15\ P + 420\ K_c = \sqrt{420\ P}\ (16 + P)$$

$$15 + 16\ P \geq 2\sqrt{420\ P}$$

The inequality is satisfied when

$$P > 4.5 \text{ or } P < 0.2$$

From the root locus consideration, a large value of P may be preferred, but too large a value tends to make the filter design impractical. This is also true with the case of two small a value of P. P = 25 is quite a good guess as is shown.

$$P = 25$$

$$B_1 = (16 + 25)\sqrt{420 \times 25} = 4182$$

$$A_1 = 15 \times 25 + 420\ K_c = 375 + 420\ K_c$$

$$K_c = 9$$

and

$$B_2 = 2\sqrt{420 \times 25} = 204$$

$$A_2 = 15 + 16 \times 25 = 415.$$

This shows the M (A_1, A_2) point surely lies in the stable domain.

The $G_c(s)$ required can thus have the following form:

$$G_u(s) = \frac{9\ (s + 2.77)}{(s + 25)}\ .$$

Example 6.6

$$G_u(s) = \frac{3780\ (s + 2.77)}{s(s + 1)(s + 15)(s + 25)}$$

is to be compensated to have $\zeta \geq 0.6$.

Solution:

If the compensator is assumed to be

$$G_c(s) = \frac{K_c(s + z)}{s + P} \quad ,$$

the characteristic equation is

$$s^5 + (41 + P)s^4 + (415 + 41P)s^3 + (375 + 415P + 3780K_c)s^2$$

$$+ (4155P + 10500K_c)s + 10500P = 0$$

The B_1 - B_2 equations are (for $\zeta = 0.6$)

$$B_1 = 1.2\ A_o/\omega + A_3\omega^2 - 1.2\ A_4\omega^3 + 0.44\ \omega^4$$

$$B_2 = A_o/\omega^2 + 1.2\ A_3\omega - 0.44\ A_4\omega^2 - 0.672\ \omega^3.$$

These equations are too complicated and it is rather unwise to have preliminary checks such as to see what ranges of P are adequate, or whether one section will suffice. It is noticed that a larger value of P is needed for the purpose from the view-point of root locus method. P = 30 is tried and the graph is depicted as in Fig. 6-8.

ω	2.5	5	7.5	10	12.5	15
B_1	16×10^4	11×10^4	11×10^4	12×10^4	13×10^4	12.8×10^4
B_2	5.4×10^4	2.2×10^4	2×10^4	1.9×10^4	2×10^4	2.1×10^4

This P = 30 is a little short of the specification as the K_c line can't hit the enclosure. The value of P must be changed a little. So the effect of changing P is checked:

$$\frac{d\ B_1}{dP} = \frac{1.2 \times 10500}{\omega} + 41.\omega_2 - 1.2\ \omega^3$$

$$\frac{d\ B_2}{dP} = \frac{10500}{\omega^2} + 1.2 \times 41\ \omega - 0.44\ \omega^2$$

This shows that at $\omega = 10$

$$B_1\ (P + \Delta P) - B_1(P) = 15500\ \Delta P$$

$$B_2\ (P + \Delta P) - B_2(P) = 157\ \Delta P$$

Also it is seen that

$$A_1(P + \Delta P, K_c + \Delta K_c) - A_1(P, K_c) = 4155\,\Delta P + 10500\,\Delta K_c$$

$$A_2(P + \Delta P, K_c + \Delta K_c) - A_2(P, K_c) = 415\,\Delta P + 3780\,\Delta K_c$$

If $K_c = 1.63$, then

$$A_1(P, K_c) = A_1(30, 1.63) = 141765$$

$$A_2(P, K_c) = A_2(30, 1.63) = 19000$$

Thus it is seen:

$$B_1(P + \Delta P) = 12 \times 10^4 + 4160\,\Delta P$$

$$A_1(P + \Delta P, K_c + \Delta K_c) = 141765 + 4155\,\Delta P + 10500\,\Delta K_c$$

$$A_2(P + \Delta P) = 1.9 \times 10^4 + 157\,\Delta P$$

$$A_2(P + \Delta P, K_c + \Delta K_c) = 19000 + 415\,\Delta P + 3780\,\Delta K_c$$

Here $B_1(P + \Delta P)$ and $B_2(P + \Delta P)$ mean B_1 and B_2 on the new curve.

To have the M (A_1, A_2) point in the enclosure the following equations are needed:

$$A_1(P + \Delta P, K_c + \Delta K_c) = B_1(P + \Delta P)$$

$$A_2(P + \Delta P, K_c + \Delta K_c) \geq B_2(P + \Delta P)$$

Substituting, the following results are obtained:

$$21765 - 11345\,\Delta P + 10500\,\Delta K_c = 0$$

$$258\,\Delta P - 3780\,\Delta K_c \geq 0.$$

If $\Delta K_c = -0.1$, then

$$\Delta P = 2$$

is obtained from the first of these equations. This makes the left hand side of the second equation equal to 894. This means the M(A_1, A_2) point lies inside the new $B_1 - B_2$ curve by the amount of 894, which assures the stability with $\zeta \geq 0.6$, as the numerical values of the graph shows.

Thus, it is seen that

$$G_c(s) = \frac{1.53\,(s + 20.9)}{(s + 32)}$$

is the required compensation.

Note: This method is applicable only when ΔP doesn't change the shape of the resulting graph too much, that is, the relative positions of maxima and minima must be kept on before and after the change of P.

6.3 Feedback Compensation

Mitrovic's method, whether it implies the original or extended one, is primarily just concerned with the system characteristic equation, and so general procedures are the same in both cascade and feedback compensation. But in the latter compensation if a pure zero filter or pure derivative, and accelerative ones are permitted, the order of the original system need not be heightened. This means some simplification.

Example 6.7

For the system of Fig. 6-9, find an adequate value for K_t to make the system have $\zeta \geq 0.6$.

Solution: The system characteristic equation is

$$s^3 + 16\, s^2 + (15 + 420\, K_t)s + 420 = 0$$

$B_o - B_1$ equations with $\zeta = 0.6$ are given to be

$$B_o = 16\, \omega^2 - 1.2\, \omega^3$$

$$B_1 = 19.2\, \omega - 0.44\, \omega^2$$

The polar analysis shows that $B_o - B_1$ curve must have the form of Fig. 6-10 to satisfy the specification. The values of ω_m are given by the tables:

$$\omega_{m1} = 8.8$$

$$\omega_{m2} = 22$$

which show the relation among the three coefficients is favorable to have an enclosure.

$$B_o(\omega_{m1}) = 668.3.$$

As A_o is kept to be 420, there must be two points on the curve for this case. $\omega = 5.9$ is found to satisfy $B_o = 420$. $B_1(\omega)$ for this value is

$$B_1(5.9) = 19.2 \times 5.9 - 0.44 \times (5-9)^2$$

$$= 98$$

$$A_1 = 15 + 420\, K_t = 98$$

Thus K_t is found to be:

$$K_t \approx 0.2$$

Example 6.8

$$G_u(s) = \frac{450\ (s + 4}{s(s + 1)(s + 5)(s + 7)}$$

is stabilized by acceleration feedback, as shown in Fig. 6-11.

What values of K_a are applicable?

Solution:

The system characteristic equation is

$$s^4 + (13 + 450K_a)s^3 + (47 + 1800\ K_a)s^2 + 485\ s + 1800 = 0$$

$B_2 - B_3$ plane is preferable in this example, which gives for $\zeta = 0$,

$$B_2 = \frac{A_o}{\omega^2} + \omega^2$$

$$B_3 = \frac{A_1}{\omega^2}$$

From the table it is seen that:

$$\omega^2_{min} = \sqrt{1800} \doteq 42.4,$$

and

$$B_2(\omega_{min}) = 84.8$$

$$B_3(\omega_{min}) = 11.4$$

From the polar analysis of stability, it is seen that the $B_2 - B_3$ curve must have the shape shown in Fig. 6-12, and M (A_2, A_3) must be inside this curve. So the following calculations are made to find the value of K_a:

ω^2	1	810	42.4	50	100	200
$\frac{1800}{\omega^2}$	1800	180	42.4	36	18	9
B_2	1801	190	84.8	86	118	209
B_3	485	48.5	11.4	9.7	4.85	2.43

The graph of Fig. 6-13 shows

$$K_a \geq \frac{17}{450} = 0.04$$

is applicable to get a stabilized system.

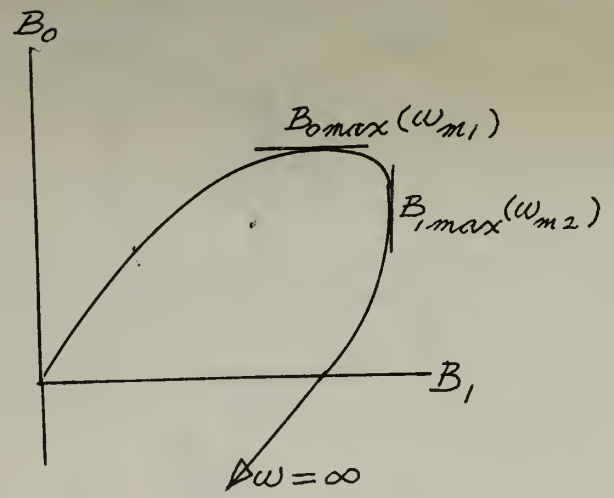


Figure 6-1

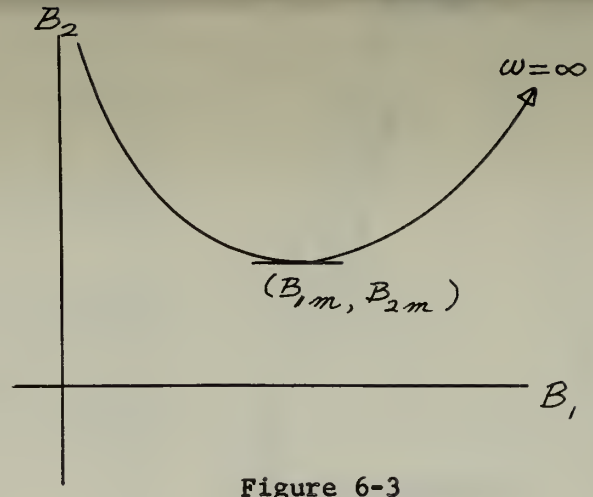
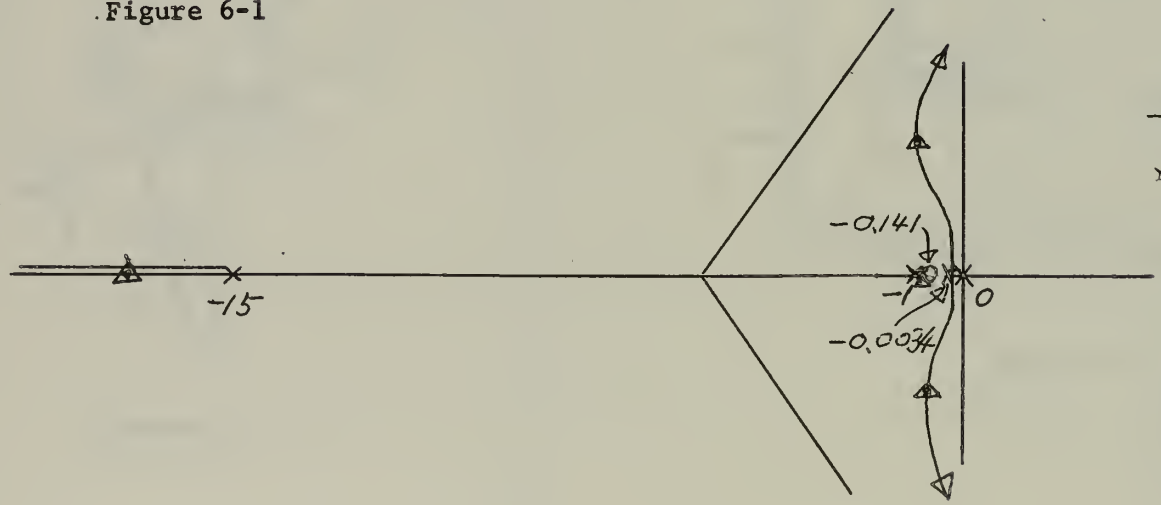


Figure 6-3



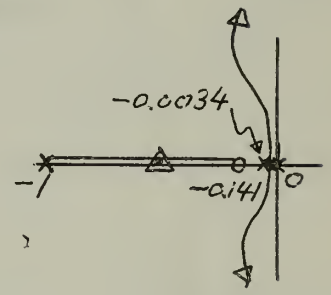


Figure 6-2

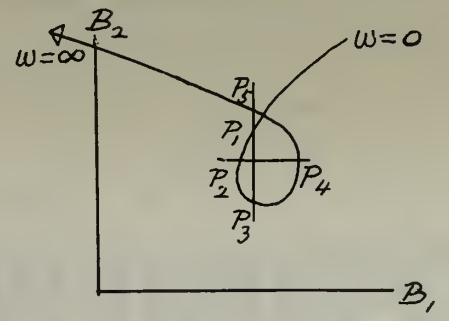


Figure 6-4

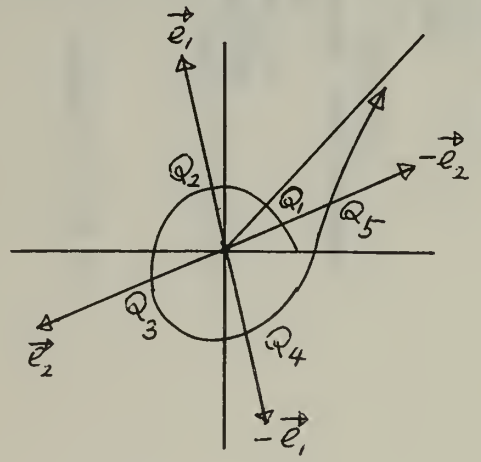


Figure 6-5

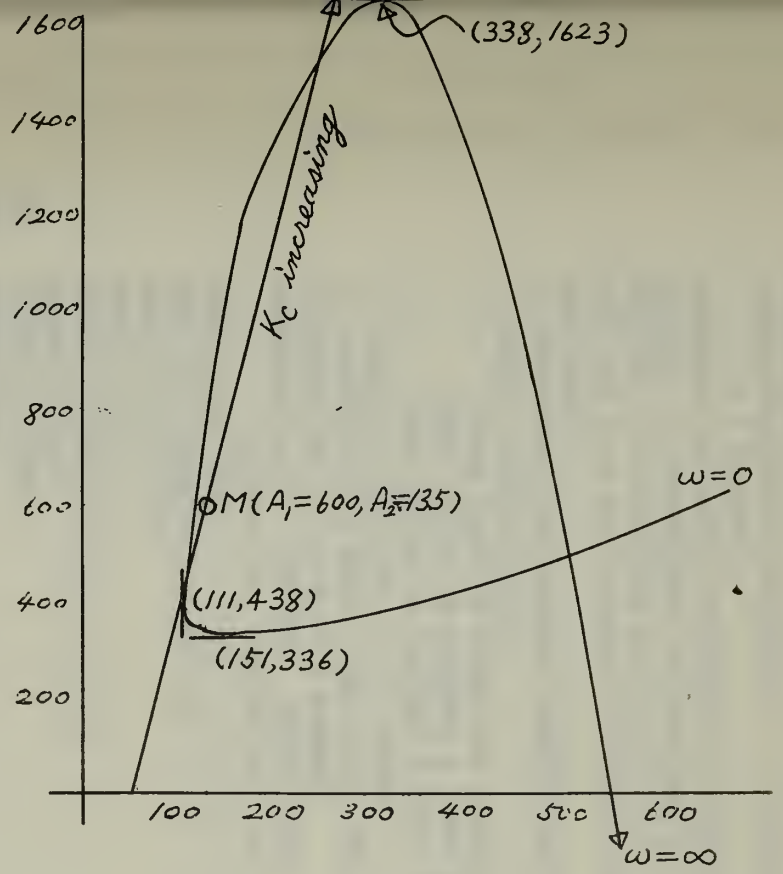
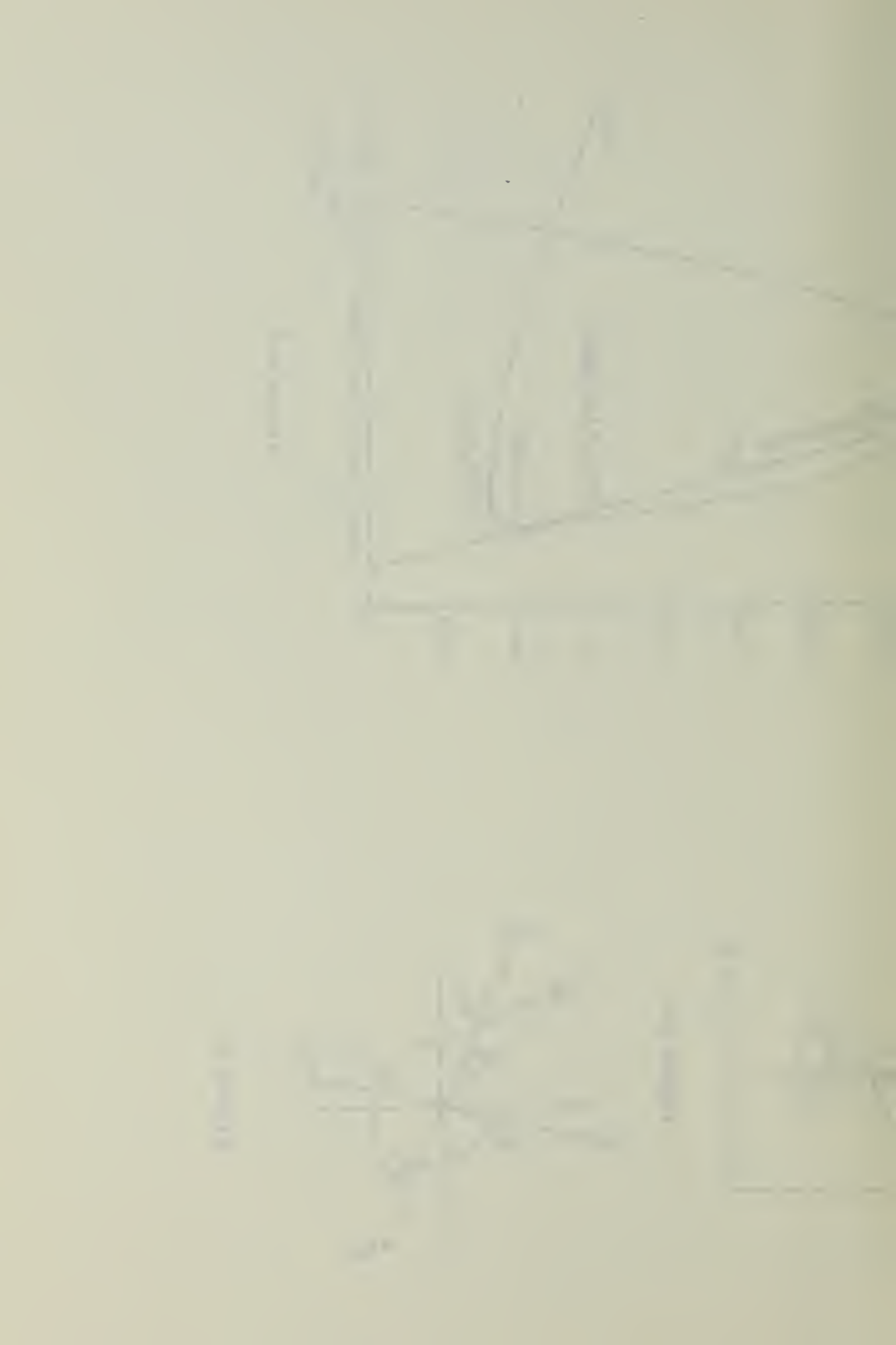


Figure 6-6



```
      PROGRAM MIT BLM
C     PROGRAM IS TO GRAPH MITROVIC THALER BL VS. BM CURVES FOR EQUATION
C     A(N)*S**N+A(N-1)* S**(N-1)+ .. + BL*S**L +..+BM*S**M+..+A(1)*
            S + AZERO=C,
C     ACCORDING TO METHOD DEVELOPED BY THALER AND OTHERS.
C     PROGRAM IS DIMENSIONED FOR MAXIMUM N = 30.
C     PROGRAM IS NORMALIZED SO AS A(N) = 1.0
C     A(L) AND A(M) ARE ASSUMED TO EXIST, BOTH BEING EQUAL TO ZERO.
C     DATA CARDS
C      1.USERS NAME AND PROGRAM IDENTIFICATION COLUMNS 1 THRU 27.
C      2.OUTPUT SPECIFICATION
C            BLANK CARD IS PRINT AND GRAPH ARE BOTH REQUIRED.
C            WORD GRAPH IN COLUMNS 1 THRU 5 IF GRAPH ONLY IS REQUIRED.
C            WORD PRINT IN COLUMNS 1 THRU 5 IF PRINT ONLY IS REQUIRED.
C      3.ORDER OF EQUATION, N, 12 FORMAT.
C      4. COEFFICIENTS AZERO THRU A(N), 8F10.2 FORMAT
C      5.VALUES OF L AND M, 212 FORMAT.
      DIMENSION A(30), PHI(60), BLAPHI (60), BMAPHI(60), BL(900), BM(900),
     1          ITITLE(10), JTITLE(10), B(31)
C     READ INPUT DATA
      READ 100,(ITITLE(J), J=1, 4)
      READ 100,IOUTPUT
      READ 101,N
      READ 102,AZERO,(A(J),J = 1,N)
      READ 103,L,M
  100 FORMAT(A7,3A8)
  101 FORMAT(12)
  102 FORMAT(8F10.2)
  103 FORMAT(212)
C     SET UP ITITLE FOR GRAPH AND JTITLE FOR PRINT.
      JTITLE(1) =8HM
      LDQ(ITITLE + 1),ENA(20B), LRS(6), STQ(ITITLE + 1)
      LDA(ITITLE + 4), ARS(6),LDQ(JTITLE + 1),LLS(6),STA(ITITLE + 4)
      ITITLE(5) =8HITROVIC
      ITITLE(6) =8HBL VS B
      ITITLE(7) =8HM CURVE
      DO 2 J=8, 10
    2 ITITLE(J) =8H
      DO 3 J =1,10
    3 JTITLE(J) = ITITLE(J)
C     WE CHECK OUTPUT REQUIREMENTS
      ITEST1 = 8H
      ITEST2 = 8HGRAPH
      ITEST3 = 8HPRINT
      IF(ITEST1 - IOUTPUT)12,11,12
   11 INDICPR = 1
      INDICGR = 1
      GO TO 17
   12 IF(ITEST2 - IOUTPUT)14,13,14
   13 INDICPR =0
      INDICGR =1
      GO TO 17
```

```
   14   IF(ITEST3-IOUTPUT)16,15,16
   15   INDICPR = 1
        INDICGR = 0
        GO TO 17
   16   PRINT 208
        STOP
  208   FORMAT (20H ERROR IN DATA CARDS)
C       PRINT RECORD OF INPUT DATA.
   17   PRINT 200
        PRINT 204,(JTITLE(J),J = 1,7)
        PRINT 206
        PRINT 207,AZERO,(A(J),J =1,N)
        PRINT 202,L
        PRINT 203,M
  200   FORMAT(1H1)
  201   FORMAT(1HO)
  204   FORMAT(4X,7A8)
  205   FORMAT(1HO,25HTHE ORDER OF EQUATION IS ,12)
  206   FORMAT(1HO,39HTHE COEFFICIENTS,AZERO THRU A(N) ARE   )
  207   FORMAT(20X,E14.6)
  202   FORMAT(7HBL IS B,12)
  203   FORMAT(7HBM IS B,12)
C       THE EQUATION IS NORMALIZED SO AS TO GET A(N) = 1.0
        ANRECIP = 1.0/A(N)
        PRINT 212
  212   FORMAT(1HO, 10X,45 HTHE RESULTS REFER TO THE NORMALIZED EQUATION.)
        AZERO = AZERO*ANRECIP
        DO 5 J=1,N
    5   A(J) = A(J)*ANRECIP
C       THE STEP SIZE WILL BE TAKEN AS 0.001 TIMES THE FREQUENCY OF INTER-
C       EST.  FOR THIS AIM WE LOOK FOR THE VALUES OF OMEGAN WHICH MAKE BOTH
        OF THE LAST TERMS IN BL AND BM DOMINATE FOR EACH VALUE OF ZETA.
        LM =L-M
        LM1 =XABSF(LM)
        IF((LM1/2)*2-LM1)175,74,75
   74   ZETA =0.1
        KK =5
        GO TO 76
   75   ZETA = 0.0
        KK =6
   76   DO 50 JJ =1,KK
C       CALCULATE CFBYSHEV POLYNOMIALS,
   20   PHI(31) =-1.
        PHI(32) =2.*ZETA
        NP30 = N+30
        DO 21 K =33, NP3C
   21   PHI(K) =-2.0*ZETA*PHI(K-1)-PHI(K-2
        PHI(30) =0.
        MN =30-N
        DO 105 J =MN,29
  105   PHI(J) =-PHI(60-J)
```

```
C       THE TERMS IN BL AND BM ARE CALCULATED IN TWO STAGES.
        DO 104 1 =1,N
  104   B(1 + 1) = A(1)
        B(1) = AZERO
        NP1 = N +1
        NM1 = N- 1
        NM2 = N-2
        L29 = 29-L
        M29 = 29-M
        DO 22 1 =1, NP1
        BLAPHI(1) = B(I) *PHI(I + M29)/PHI(30-LM)
   22   BMAPHI(I) = B(I) *PHI(I + L29)/PHI(30+LM)
        OMEGAM = 0.5
        DO 8K = 1,12
        OMEGAM = 2.0*OMEGAM
        SL = BLAPHI(1) * OMEGAM**(-L)
        L1 = L + 1
        DO 311 1 = 2,NP1
  311   SL = SL + BLAPHI(I)*OMEGAM**(I -L1)
        NL =N-L
        IF(BLAPHI(NP1))71,170,71
  170   IF(BLAPHI(N))7C,172,70
   71   BSA =BLAPHI(NP1)*OMEGAM**NL
        GO TO 72
   70   BSA = BLAPHI(N)*OMEGAM**(NL - 1)
        GO TO 72
  172   IF(BLAPHI(NM1))171,173,171
  171   BSA = BLAPHI(NM1)*OMEGAM**(NL -2)
        GO TO 72
  173   BSA = BLAPHI(NM2)*OMEGAM**(NL - 3)
   72   BSLN = ABSF(BSA)
        BSL =ABSF(SL-BSA)
        IF(BSL-BSLN)9,8,8
    8   CONTINUE
    9   STEP = OMEGAM
        OMEGAN = 0.5
        DO 68 K =1,12
        OMEGAN = OMEGAN*2.0
        SM =BMAPHI(1)*OMEGAN**(-M)
        M1 = M + 1
        DO 312 1 =2,NP1
  312   SM = AM +BMAPHI(I) * OMEGAN**(I-M1)
        IF(BMAPHI(NP1))81,180,81
  180   IF(BMAPHI(N))80, 182,80
   81   BSB = BMAPHI(NP1)*OMEGAN**NM
        GO TO 82
   80   BSB = BMAPHI(N)*OMEGAN**(NM-1)
        GO TO 82
  182   IF(BMAPHI(NM1))181,183,181
  181   BSB = BMAPHI(NM1)*OMEGAN**(NM-2)
        GO TO 82
  183   BSB =BMAPHI(NM2)*OMEGAN**(NM-3)
```

```
 82   BSMN = ABSF ( BSB)
      BSM = ABSF (SM - BSB)
      IF (BSM - BSMN)69,68,68
 68   CONTINUE
 69   STEN =OMEGAN
      IF(STEP - STEN)216,215,215
216   STEP = STEN*0.001
      GO TO 123
215   STEP =STEP*0.001
123   DO 310 IOMEGAN =1,900
      OMEGAN =IOMEGAN
      OMEGAN =OMEGAN*STEP
      SUML =BLAPHI (1) *OMEGAN**(-L)
      SUMM =BMAPHI (1) *OMEGAN**(-M)
      DO 23 I=2,NP1
      SUML=SUML +BLAPHI(I)*OMEGAN**(I-L1)
 23   SUMM=SUMM +BMAPHI(I)*OMEGAN**(I-M1)
      BL(IOMEGAN) = SUML
      BM(IOMEGAN) = SUMM
C     PRINT EVERY TENTH POINT IF PRINT OUT IS REQUIRED.
      IF(INDICPR-1) 310,24,310
 24   IF(OMEGAN-1) 25,27,25
 25   IF(XMODF(LOMEGAN , 100))26,29,26
 26   IF(XMODF(IOMEGAN ,  10))310,30,310
 27   PRINT 200
      PRINT 209,ZETA
      PRINT 210
      GO TO 310
 29   PRINT 201
 30   PRINT 211,OMEGAN,BL(IOMEGAN),BM(IOMEGAN)
209   FORMAT(10X,35HRESULTS OF COMPUTATION WITH ZETA =  ,F4.2)
210   FORMAT(7//20X, 10H  OMEGAN  ,10H   BL    ,10H   BM     //)
211   FORMAT(19X,E11.5,2X,E11.5,2X,E11.5)
      CONTINUE
      NUMPTS = 900
C     GRAPH IF GRAPH IS REQUIRED
      IF(INDICGR -1)49,132,49
132   IF((LM1/2)*2-LM1)134,232,134
134   GO TO (33,34,35,36,37,38),JJ
232   GO TO (33,34,35,36,37),JJ
 33   MODCURV =0
      LABEL = 2H1
      GO TO 39
 34   MODCURV =0
      LABEL =2H2
      GO TO 39
 35   MODCURV =0
      LABEL = 2H3
      GO TO 39
 36   MODCURV =0
      LABEL = 2H4
      GO TO 39
 37   MODCURV =0
```

```
      LABEL =2H5
      GO TO 39
38    MODCURV =0
      LABEL =2H6
39    SFX =0.0
      SFY =0.0
      MINOFFX =0
      MINOFFY =0
      LABELNO =11
      MODE = 0
      N1 =0
      NE =0
      CALL GRAPH2 (NUMPTS,BL,BM,8,MODCURV,LABEL,ITITLE,SFX,SFY,MINOFFX,
     IMINOFFY,LABELNO,MODE,N1,N2)
49    ZETA =ZETA +0.2
50    CONTINUE
      END
      END
```

APPENDIX

If ω_n goes to zero in the equation (3-5),

$$\vec{F}(\omega_n) = (A_i - B_i)\,\omega_n^i\,\vec{e}_i + (A_j - B_j)\,\omega_n^j\,\vec{e}_j \qquad (3\text{-}5)$$

it becomes

$$\lim_{w_n \to 0} \vec{F}(\omega_n) = -\lim \left\{ B_i(\omega_n^i\,\vec{e}_i + B_j(\omega_n)\,\omega_n^j\,\vec{e}_j \right\}$$

$$= A_o\left[\frac{\varphi_j(\zeta)}{\varphi_{i-j}(\zeta)}\,\vec{e}_i - \frac{\varphi_i(\zeta)}{\varphi_{i-j}(\zeta)}\,\vec{e}_j\right]$$

$$= \frac{A_o}{\varphi_{i-j}(\zeta)}\left[\varphi_j(\zeta)\,\vec{e}_i - \varphi_i(\zeta)\,\vec{e}_j\right] \qquad (A\text{-}1)$$

Here $\vec{e}_i$ and $\vec{e}_j$ are

$$\vec{e}_i = \underline{/\ i\,\frac{\pi}{2} + i\theta} = (-\zeta + I\sqrt{1-\zeta^2})^i$$

$$\vec{e}_j = \underline{/\ j\,\frac{\pi}{2} + j\theta} = (-\zeta + I\sqrt{1-\zeta^2})^j \qquad (A\text{-}2)$$

and the following expressions are derived

$$\vec{e}_1 = -\zeta + I\sqrt{1-\zeta^2} = \varphi_o(\zeta) = \zeta\,\varphi_1(\zeta) - I\sqrt{1-\zeta^2}\,\varphi_1(\zeta)$$

$$\vec{e}_2 = -1 + 2\zeta^2 + I\sqrt{1-\zeta^2}\,(\zeta) = \varphi_1(\zeta) + \zeta\,\varphi_2(\zeta) -$$

$$-I\sqrt{1-\zeta^2}\,\varphi_2(\zeta)$$

.....

$$\vec{e}_n = \varphi_{n-1}(\zeta) + \zeta\,\varphi_n(\zeta) - I\sqrt{1-\zeta^2}\,\varphi_n(\zeta). \qquad (A\text{-}3)$$

From these formula, it is seen that

$$\varphi_i(\zeta)\,\vec{e}_j = \varphi_i(\zeta)\,\varphi_{j-i}(\zeta) + \zeta\,\varphi_i(\zeta)\,\varphi_j(\zeta) -$$

$$- I\sqrt{1-\zeta^2}\,\varphi_i(\zeta)\,\varphi_j(\zeta) \qquad (A\text{-}4)$$

$$\varphi_j(\zeta)\,\vec{e}_i = \varphi_j(\zeta)\,\varphi_{i-1}(\zeta) + \zeta\,\varphi_j(\zeta)\,\varphi_i(\zeta) -$$

$$- I\sqrt{1-\zeta^2}\,\varphi_j(\zeta)\,\varphi_i(\zeta)$$

APPENDIX (Continued)

Subtracting the latter from the former, it is seen

$$\varphi_i(\zeta)\,\vec{e}_j - \varphi_j(\zeta)\,\vec{e}_i = \varphi_i(\zeta)\,\varphi_{j-1}(\zeta) - \varphi_j(\zeta)\,\varphi_{i-1}(\zeta) \qquad \text{(A-7)}$$

Also it is seen that

$$\varphi_i(\zeta)\,\varphi_{j-1}(\zeta) - \varphi_j(\zeta)\,\varphi_{i-1}(\zeta) = \varphi_{i-j}(\zeta) \qquad \text{(A-8)}$$

from a simple mathematical induction.

Substituting (A-7) and (A-8) into (A-1) gives

$$\lim_{\omega_n \to 0} \vec{F}\,(\omega_n) = A_o$$

which is the required result, equivalent to equation (3-6)

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